Maximal subgroups of classical groups in dimensions 16 and 17

by

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Thesis

Submitted to the University of Warwick

for the degree of

Doctor of Philosophy

Department of Mathematics

February 2017

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Acknowledgments

First and foremost, I would like to thank my supervisor Professor Derek Holt for his guidance and support throughout my studies. This thesis would not have been possible without his expert guidance, diligent proofreading and generosity of time throughout, as well as his patience and persistence in helping me to grow as a mathematician. I am also immensely grateful to Professor Don Taylor for his invaluable suggestions whch formed the basis of my approach in Chapter 5. I am also grateful to the many people with whom I have discussed the ideas and problems within this thesis; particular thanks to Gareth Tracey for many useful conversations and comments on my work. My thanks also to EPSRC for funding my studies.

I have been very fortunate during my time at Warwick to work in a friendly and supportive environment. I am grateful for all those I have interacted with, be they students, academic or non-academic staff. My particular thanks go to Alex Bartel, Mark Bell, Chris Birkbeck, Florian Bouyer, Mark Cummings, Rhiannon Dougall, Carole Fisher, George Kenison, Ronja Kuhne, Nicolas Mascot, Dave McCormick, Vandita Patel, Ben Pooley, Matthew Spencer, Alex Torzewski, Katie Vokes and Alex Wendland; my thanks and apologies also go to those who should also be on that list whom I have inadvertently forgotten.

My thanks also go to my friends Emma, Steve, Chaoyi, Sarah, Dave, Andrew, Lynn, Owen, Geraint and Barney. I value and appreciate the friendship I have with each of you, the support I have received from each of you throughout my PhD, and the numerous evenings and weekends spent talking, relaxing and playing board games.

I am very grateful for all those at St Paul's, Leamington Spa - it has been a pleasure and a joy to be welcomed into such a friendly and loving community. Particular thanks go to my small group, with whom I shared my highs, my lows and my favourite motorway service stations.

Last but by no means least, my thanks to my family for their love and support, and especially to my parents for the innumerable ways they have supported me unwaveringly in everything I have done.

Declarations

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree.

The work presented was carried out by the author, unless otherwise referenced.

Abstract

Aschbacher's Theorem [1] subdivides maximal subgroups of the classical groups and their almost simple extensions into nine classes, denoted $\mathcal{C}_1, \ldots, \mathcal{C}_9$. The first eight of these classes contain the so-called 'geometric-type' subgroups. Members of these classes have been classified fully for classical groups in dimensions 13 and higher in [33], and in the low-dimensional case in [8]. Class $\mathcal{C}_9$, or $\mathcal{S}$, consists of groups which are almost simple modulo their centre. There is currently no description of all members of this class in all dimensions. In [8], the authors describe all members of class $\mathcal{S}$ in dimensions up to 12, and in [48] the author describes these in dimensions $13 - 15$. In this thesis, we will extend these results to determine the members of class $\mathcal{S}$ in dimensions 16 and 17 (and thus all maximal subgroups of classical groups in these dimensions) except in the case of the orthogonal groups where some results are conjectured.

Chapter 1 provides background information, including the subdivision of class $\mathcal{S}$ into subclasses $\mathcal{S}_1$ and $\mathcal{S}_2$. Chapters 3 and 4 describe the members of $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively for 16- and 17-dimensional classical groups, and Chapter 6 describes containments between these classes. The list of maximal subgroups is summarised in Chapter 7.

We also provide some general results which can be applied to members of class $\mathcal{S}$ in classical groups of other dimensions. Chapter 2 discusses results which can be applied to a class of $\mathcal{S}_2$-candidate subgroups whose field automorphism is induced by a permutation matrix. In Chapter 5 we provide a construction of the natural representation of the spin and half-spin groups and their normalisers.

Chapter 1

Background

1.1 Introduction

The purpose of this thesis is to prove the following theorem.

Theorem 1.1.1. *Let q be a prime power, let $n \in \{16, 17\}$ and let Ω be a quasisimple group equal to one of $\mathrm{SL}_n(q), \mathrm{SU}_n(q), \mathrm{Sp}_n(q)$ or $\Omega_n^\epsilon(q)$. Let $\overline{G}$ be an almost simple extension of $\overline{\Omega} = \Omega/Z(\Omega)$. Further, if $\Omega = \Omega_n^\epsilon(q)$ with $(n, \epsilon) \in \{(16, +), (17, \circ)\}$, assume additionally that Conjecture 2.3.3 holds and that $G < \mathrm{CGO}_n^\epsilon(q)$. Then representatives of the conjugacy classes of maximal subgroups of $\overline{G}$ that do not contain $\overline{\Omega}$ are as specified in the appropriate table in Chapter 7.*

In the case where $\Omega = \Omega_n^\epsilon(q)$, we can also describe precisely the situations where the structure of maximal subgroups is conjectured.

- For $\Omega = \Omega_{17}^\circ(p^2)$ when $p \equiv \pm 2 \mod 5$, the class stabiliser of the $\mathcal{S}_1$-candidate subgroup $\mathrm{L}_2(16)_2$ in the full automorphism group of Ω is conjectured to be $\langle \phi \rangle$ when $p \equiv \pm 1 \mod 8$ and $\langle \phi\delta \rangle$ when $p \equiv \pm 3 \mod 8$.
- For $\Omega = \Omega_{16}^+(q)$, we have:
  - The field automorphism of order 4 of the $\mathcal{S}_2$-candidate subgroup $G = \mathrm{L}_2(q^4).4$ is realisable over Ω if Conjecture 2.3.3 is true; otherwise, it is also possible that the field automorphism of order 4 is induced by δ' and the $\mathcal{S}_2$-candidate subgroup is $\mathrm{L}_2(q^4).2$. It is also conjectured that the field automorphism ϕ_Ω of Ω induces the field automorphism ϕ_G of G.
  - Similarly, the field automorphism of order 2 of the $\mathcal{S}_2$-candidate subgroup $\mathrm{S}_4(q^2).2$ is realisable over Ω if Conjecture 2.3.3 is true; otherwise, it is also possible that the field automorphism of order 2 is induced by δ' and

the $\mathcal{S}_2$-candidate subgroup is $\mathrm{S}_4(q^2)$. It is also conjectured that the field automorphism ϕ_Ω of Ω induces the field automorphism ϕ_G of G.

To prove Theorem 1.1.1, we will follow closely the approach of the authors in [8]. The main focus of this thesis will be to classify the groups in Class $\mathcal{C}_9$, otherwise known as $\mathcal{S}$, in Aschbacher's theorem, since groups in the remaining Aschbacher classes (the geometric-type subgroups) have been determined in [33].

In the remainder of this chapter, we will introduce aspects of the theory that we will require for the computations in later chapters, including defining the terms used in Theorem 1.1.1. We will provide an introduction to classical groups, as well as Aschbacher's theorem. We will also introduce the results we need regarding simple groups, finite fields, representation theory and number theory.

In Chapter 2, we introduce a family of members of Class $\mathcal{S}_2$ known as rewritten tensor field groups. Such groups have a field automorphism which is induced by a conjugate of a permutation matrix, and in this chapter we compute the determinant and (assuming an additional conjecture) spinor norm/quasideterminant of such permutation matrices with respect to a particular form in full generality.

In Chapter 3, we classify the $\mathcal{S}_1$-candidates, one of the sub-classes of the Aschbacher class $\mathcal{S}$. We first describe the general approach and theory as given in [8, Chapter 4], before describing in detail the candidate $\mathcal{S}_1$-maximal subgroups in dimensions 16 and 17 (for expository purposes we provide the computations in dimension 17 first, since these are generally more straightforward).

In Chapter 4, we perform similar computations for the $\mathcal{S}_2$-candidates. We again provide an introduction to the theory required following [8, Chapter 5], including a brief introduction to highest weight theory, before again giving a detailed construction of all the candidate $\mathcal{S}_2$-maximal subgroups.

In Chapter 5, we provide an explicit construction of the spin and half-spin representations and their extensions by automorphisms, by building these representations from the corresponding spin and half-spin representations of smaller groups, and verifying correctness using the Curtis-Steinberg-Tits presentation. We consider these groups as $\mathcal{S}_2$-candidate subgroups of certain classical groups and perform the computations as in Chapter 4 in a more general setting. We also provide a MAGMA implementation of this construction.

In Chapter 6, we check containments between the geometric-type subgroups and the groups in classes $\mathcal{S}_1$ and $\mathcal{S}_2$. Finally, in Chapter 7 we summarise the maximal subgroups.

1.2 Notation

The notation we use will be consistent with [8], and we refer the reader to [8, Section 1.2] for a full description of the standard notation required. We will introduce any non-standard notation as it is required.

For a group G and group elements $g, h \in G$, we write $g^h = h^{-1}gh$ and $[g, h] = g^{-1}h^{-1}gh$.

Throughout, all vectors will be row vectors and matrices will typically act on vector spaces via a right action.

For integers a and b, we denote $a \mod b$ by $a(b)$.

By $H < G$, we mean that H is a subgroup of G, allowing the possibility that $H = G$. If we explicitly wish to rule out the latter possibility we will use $H \lneq G$.

Our notation for groups, especially simple groups and group extensions, follow the ATLAS notation as given in [12]. In particular, A_n and S_n will refer to the alternating and symmetric group respectively. Sometimes we will consider A_n and S_n as acting on a particular set X; when this is important, we will use the alternative notation $\mathrm{Alt}(X)$ and $\mathrm{Sym}(X)$. In the special case where $X = \{1, \ldots, n\}$, the natural A_n- or S_n-set, we will denote these groups by $\mathrm{Alt}(n)$ and $\mathrm{Sym}(n)$ respectively, to emphasise that we are considering the natural permutation action of these groups.

There may be some possibility of confusion between notation for the alternating group on n points and the Dynkin diagram for $\mathrm{SL}_{n+1}(q)$; to avoid this, we will denote simple Lie algebras in gothic font, for instance $\mathfrak{A}_n$.

1.3 First definitions and motivation

Definition 1.3.1. A subgroup M of a group G is *maximal* if $M \neq G$ and there exists no other subgroup $N < G$ such that $M \lneq N \lneq G$.

Understanding maximal subgroups is useful for a number of reasons. We will briefly illustrate two.

Firstly, there is the following obvious consequence of the definition, which nonetheless is useful to state.

Lemma 1.3.2. *Let $S < G$ be any subgroup, with $S \neq G$. Then $S < M$ for some maximal subgroup M of G.*

Thus, understanding all subgroups of G is equivalent to understanding all subgroups of maximal subgroups of G.

The second application is to understanding permutation representations of groups.

Definition 1.3.3. Let G act on a set X, with the action of $g \in G$ on $x \in X$ denoted by x^g.

- The action is *transitive* if, for every $x, y \in X$, there exists an element $g \in G$ such that $x^g = y$.
- A *block* for the action is a nonempty subset $B \subset X$ such that for any $g \in G$, either $B^g = B$ or $B^g \cap B = \emptyset$.
- The action is *primitive* if it is transitive and the only blocks B of the action satisfy either $B = X$ or $|B| = 1$.

Lemma 1.3.4. *[46, Theorem 9.15] Let G act transitively on a set X. Then the action of G on X is primitive if and only if for every $x \in X$, the stabiliser $\mathrm{Stab}_G(x) = \{g \in G | x^g = x\}$ is a maximal subgroup of G.*

An immediate consequence of the above lemma is that, for a given group G, there are only finitely many sets X that G acts upon primitively (up to equivalence). Indeed, there is a one-to-one correspondence between primitive permutation representations and conjugacy classes of maximal subgroups, since if we are given a maximal subgroup $M < G$, G acts primitively on the set of cosets $\{Mg : g \in G\}$, a set of size $|G : M|$. Hence, we can understand all primitive permutation representations of G by understanding its maximal subgroups.

1.4 Simple groups

Finding all maximal subgroups of classical groups relies heavily on understanding simple groups. In this section we provide a very brief introduction to some of the theory we will require. For a more detailed introduction to the simple groups, see [59].

Definition 1.4.1. A group G is *simple* if it has precisely two normal subgroups; 1 and G.

Simple groups are of interest since every group can be said to be "built" from simple groups.

Definition 1.4.2. A *maximal normal subgroup* H of a group G is a normal subgroup of G such that $H \neq G$ and there are no other normal subgroups N of G such that $H \lneq N \lneq G$; equivalently, such that G/H is simple.

A *composition series* for G is a series of subgroups

$$G = G_0 > G_1 > \ldots > G_r = 1$$

where each G_{i+1} is a maximal normal subgroup of G_i.

The *composition factors* of a group G are the simple groups

$$G_0/G_1, \ldots, G_{r-1}/G_r.$$

A priori the composition factors of a group may appear to depend on the choice of composition series; however the next theorem shows that this is not the case.

Theorem 1.4.3 (Jordan-Hölder Theorem)**.** *[46, Theorem 5.12] Let G be a group, and suppose that we have two composition series for G, namely $G = G_0 > G_1 > \ldots > G_r = 1$ and $G = H_0 > H_1 > \ldots > H_s = 1$. Then $r = s$ and there is a bijection between the multisets of simple groups $\{G_i/G_{i+1}\}$ and $\{H_i/H_{i+1}\}$.*

Hence to each group G we can associate a list of simple groups, namely its composition factors. The converse is not true; the composition factors of a group do not generally determine G. The question of determining all groups with a certain set of composition factors is known as the extension problem and is an open problem.

A very important result, and one that plays a crucial part in many areas of finite group theory, is the classification of finite simple groups (CFSG). The classification was a collaborative effort by hundreds of mathematicians over several decades and thousands of journal pages.

Theorem 1.4.4 (CFSG)**.** *Let G be a finite simple group. Then G is isomorphic to one of the following groups:*

- *C_p, a cyclic group of prime order;*
- *A_n, the alternating group on $n \geq 5$ points;*
- *a group of Lie type; or*
- *one of 26 sporadic simple groups.*

The unique nonabelian simple composition factor of the classical groups are examples of simple groups of Lie type, and will be the main area of focus for us. See [59, Chapter 2] for more information on the alternating groups, [59, Chapter 4] for information on the exceptional groups of Lie type, and [59, Chapter 5], [19, Chapter 5.3] or [21] for more information on the sporadic groups.

We will also provide two further definitions closely related to simplicity in the following sections.

1.4.1 Quasisimple groups and the Schur multiplier

Definition 1.4.5. A group G is *perfect* if it has no nontrivial abelian quotients (equivalently, G is equal to its commutator subgroup $[G, G]$).

A group G is *quasisimple* if it is perfect, and $G/Z(G)$ is a nonabelian simple group. In other words, G is a perfect central extension of a nonabelian simple group.

We will follow the notational standard of the Atlas [12] and denote the quasisimple group G by $Z^{\cdot}S$, where $Z = Z(G)$ and $G/Z = S$, where S is a nonabelian simple group.

Closely related to the notion of a quasisimple group is the Schur multiplier.

Definition 1.4.6. Let G and K be groups, with K abelian. Then a *stem extension of G by K* is a group C with a normal subgroup K such that $C/K \cong G$ and $K < Z(C) \cap [C, C]$.

The following theorems are well-known.

Theorem 1.4.7 (Schur, 1904)**.** *[46, Theorem 7.66] For any group G, there exists a largest group $M(G)$ such that a stem extension of G by $M(G)$ exists.*

Theorem 1.4.8. *[46, Theorem 11.11, Corollary 11.12] If G is perfect, then the stem extension of G by $M(G)$ is uniquely defined up to isomorphism, and any stem extension of G by any other group K is a quotient of this extension by a subgroup of $M(G)$. Further, the stem extension of G by $M(G)$ is also perfect.*

Definition 1.4.9. The group $M(G)$ as given in Theorem 1.4.7 is the *Schur multiplier* of G.

Remark 1.4.10. The Schur multiplier is defined formally using homological algebra; see for instance the definition in [46, Chapter 7, p.201], although we will not require this definition in this thesis.

If S is a nonabelian simple group, then applying Theorem 1.4.8 gives us that there exists a largest quasisimple group whose nonabelian composition factor is S.

The Schur multipliers of the finite simple groups are known; see [19, Theorem 5.2.3] for Schur multipliers of the alternating groups, [19, Section 6.1] for Schur mulitipliers of the groups of Lie type (including the classical groups) and [19, Section 5.3] or [12] for Schur multipliers of the sporadic simple groups.

For general finite groups G, stem extensions of G by $M(G)$ are only defined up to isoclinism, as defined in Section 1.4.3.

1.4.2 Almost simple groups

Note that if S is a nonabelian simple group, then we have an isomorphism $S \cong \mathrm{Inn}(S)$, the group of inner automorphisms of S induced by conjugation by elements of S. Hence we can consider S as a subgroup of its full automorphism group $\mathrm{Aut}(S)$.

Definition 1.4.11. A group G is *almost simple* if there exists a nonabelian simple group S such that $S \leq G \leq \mathrm{Aut}(S)$.

We again follow the notational convention of the Atlas. If S is a simple group and $H < \mathrm{Out}(S)$, then we denote an extension of H by S (i.e. a group G with a normal subgroup S such that $G/S \cong H$) by $S.H$, and will refer to G as an *almost simple extension of* S. If H is generated by a single automorphism α we will often denote this group by $S.\alpha$.

Often we will be considering groups G such that $G/Z(G)$ is almost simple; alternatively, we can think of G as having a normal subgroup H which is quasisimple. Since H is not simple it would not be correct to refer to G as an almost simple extension of H; instead we will refer to G as an *extension of* H *by automorphisms.*

1.4.3 Isoclinism

Here, we follow the definitions as given in [12, Section 6.7].

Definition 1.4.12. Two groups G and H are *isoclinic* if there are isomorphisms between $G/Z(G)$ and $H/Z(H)$, and between $[G,G]$ and $[H,H]$, which commute with the commutator map.

Remark 1.4.13. Equivalently, G and H are isoclinic if there exists a larger group K such that $G < K$, $H < K$ and K is generated by $Z(K)$ and G, and also by $Z(K)$ and H. In other words, we can enlarge the centres of G and H to get isomorphic groups.

Example 1.4.14.

1. All abelian groups are isoclinic; indeed, if G and H are abelian then $G/Z(G)$, $H/Z(H)$, $[G,G]$ and $[H,H]$ are all trivial. For the alternate characterisation as given in Remark 1.4.13, G and H are both subgroups of the abelian group $G \times H$.

2. D_8 and Q_8 are isoclinic. Indeed, take i to be an element of a field such that $i^2 = -1$, and let $g_1 = \mathrm{diag}(i, -i)$ and g_2 be the permutation matrix given by the permutation $(1, 2)$. Then $\langle g_1, g_2 \rangle \cong D_8$ and $\langle g_1, ig_2 \rangle \cong Q_8$, and both are subgroups of the group of order 16 generated by $\langle g_1, g_2, iI \rangle$.

3. Suppose G is a nonabelian simple group with Schur multiplier of even order and an outer automorphism of order 2, and suppose we have a faithful representation of the corresponding group $H^+ = 2^{\cdot}G.2$. Suppose as before that i is an element of the field such that $i^2 = -1$. Then H^+ is generated by $2^{\cdot}G$ and a matrix $h \in H^+ \setminus 2^{\cdot}G$. There is another extension generated by $2^{\cdot}G$ and ih. This group, which we denote H^-, is isoclinic to H^+ as both embed inside the group generated by $2^{\cdot}G, h$ and iI, although H^+ and H^- are not generally isomorphic. For groups such as these, the character tables in [12] record only one of these groups. For instance, the character χ_9 of $2^{\cdot}A_5.2$ as given in [12, p. 2] gives rise to a representation of $2^{-\,\cdot}A_5.2$ by multiplying the character values on $2^{\cdot}A_5.2 \setminus 2^{\cdot}A_5$ by i_2, giving a character which takes values r_2 on elements in the conjugacy class $4A$. Thus, when considering representations of such groups we will need to consider the possibility of isoclinic variants.

1.5 Finite fields

We provide a brief introduction to the aspects of finite field theory we will need for this thesis. We will use these results without further reference throughout.

The classification of all finite fields is well-known; see for instance [11, Theorem 7.8.2] for a proof of the following theorem.

Theorem 1.5.1 (Moore, 1893). *For each prime p and each $e \geq 1$ there is exactly one field of $q = p^e$ elements (up to isomorphism), and these are the only finite fields.*

For example, the finite field $\mathbb{F}_p$ of order p is $\mathbb{Z}/p\mathbb{Z}$. More generally, one can obtain a finite field of order p^e via the quotient of the polynomial ring $\mathbb{F}_p[x]$ by the ideal generated by an irreducible polynomial of degree e. We will denote the finite field of order q by $\mathbb{F}_q$.

Definition 1.5.2. For a field $\mathbb{F}$, $\mathbb{F}^* = \mathbb{F} \setminus \{0\}$ denotes the set of units of $\mathbb{F}$, and is an abelian group under multiplication.

Theorem 1.5.3. *[11, Theorem 7.7.3] Let $q = p^e$ for p prime. Then every element x of $\mathbb{F}_q$ satisfies $x^q = x$, and the multiplicative group $\mathbb{F}_q^*$ is cyclic of order $q - 1$.*

Definition 1.5.4. Let ν denote a generator of the cyclic group $\mathbb{F}_q^*$. Then ν is said to be a *primitive element* of $\mathbb{F}_q^*$.

We will also refer to ν as a primitive element of $\mathbb{F}_q$.

Lemma 1.5.5. *Let e, f be positive integers such that $f|e$, and let ν denote a primitive element of $\mathbb{F}_{p^e}$. Then there is an embedding of $\mathbb{F}_{p^f}$ into $\mathbb{F}_{p^e}$, with $\mathbb{F}_{p^f}$ generated by $\nu^{\frac{p^e-1}{p^f-1}}$, which is a primitive element of $\mathbb{F}_{p^f}$.*

Proof. It is straightforward to check that the element $\nu^{\frac{p^e-1}{p^f-1}}$ has order $p^f - 1$ and generates a field of order p^f, which by uniqueness of finite fields must be $\mathbb{F}_{p^f}$. □

At various stages we will need to consider whether an element $x \in \mathbb{F}_q$ is an n-th power of an element of $\mathbb{F}_q$.

Definition 1.5.6. Let n and q be integers. Write $n = q^\alpha t$ with $q \nmid t$. Then α is said to be the *q-part of n*, and is denoted $\nu_q(n)$.

Lemma 1.5.7. *Let ν be a primitive element of $\mathbb{F}_q$. Then ν^k has odd order if and only if $\nu_2(q-1) \leq \nu_2(k)$.*

Proof. $(\nu^k)^{\frac{q-1}{(q-1,k)}} = \nu^{\mathrm{lcm}(q-1,k)} = 1$, and by definition of the lowest common multiple this is the smallest such power; hence ν^k has order $\frac{q-1}{(q-1,k)}$. The result follows. □

Lemma 1.5.8. *Let ν be a primitive element of $\mathbb{F}_q$. Then ν^k is a 2^n-th power if and only if $\min\{n, \nu_2(q-1)\} \leq \nu_2(k)$.*

Proof. Clearly if $2^n|k$ then ν^k is expressible as a 2^n-th power. Otherwise, if ν^k has odd order then it generates a multiplicative subgroup of $\mathbb{F}_q^*$ of odd order, and the squaring map is an automorphism on such a group. Hence if ν^k has odd order, it is a 2^r-th power for any $r \geq 1$. By Lemma 1.5.7 ν^k has odd order if and only if $\nu_2(q-1) \leq \nu_2(k)$. Hence ν^k is a 2^n-th power if $n \leq \nu_2(k)$ or $\nu_2(q-1) \leq \nu_2(k)$. Conversely, if neither of the previous conditions hold then $\nu^{k/\nu_2(k)}$ has even order and is raised to an odd power; hence it cannot be a square and so ν^k cannot be a 2^n-th power. □

The automorphism group of $\mathbb{F}_{p^e}$ is also easy to determine.

Definition 1.5.9. The map $\mathbb{F}_{p^e} \to \mathbb{F}_{p^e}$ mapping $a \mapsto a^p$ is called the *Frobenius mapping.*

Lemma 1.5.10. *[11, Theorem 7.8.3] The automorphism group of $\mathbb{F}_{p^e}$ is cyclic of order e, and is generated by the Frobenius mapping.*

1.6 Classical groups

In this section, we will introduce the various types of classical group, and their outer automorphism groups.

We first introduce some notation. If A is an $a \times b$ matrix, with (i,j)-th entry given by $a_{i,j}$, we will write $A = (a_{i,j})_{a\times b}$. If a and b are clear from the context we will just write $A = (a_{i,j})$.

By $E_{i,j}^{(n)}$, we mean the $n \times n$ matrix with a 1 in the (i,j)-th entry and 0 everywhere else. If n is clear from the context we will write $E_{i,j}$.

Throughout this section, let $q = p^e$ be a prime power.

1.6.1 Direct and anti-direct products

The following notation will be used frequently, especially when considering forms.

Definition 1.6.1. A *diagonal matrix*, denoted $\mathrm{diag}(a_1, \ldots, a_n)$, has the form

$$\begin{bmatrix} a_1 & & & & \\ & a_2 & & \text{\Large 0} & \\ & & \ddots & & \\ & \text{\Large 0} & & a_{n-1} & \\ & & & & a_n \end{bmatrix}.$$

An *anti-diagonal matrix*, denoted $\mathrm{antidiag}(a_1, \ldots, a_n)$, has the form

$$\begin{bmatrix} & & & & a_n \\ & \text{\Large 0} & & a_{n-1} & \\ & & \iddots & & \\ & a_2 & & \text{\Large 0} & \\ a_1 & & & & \end{bmatrix}.$$

Definition 1.6.2. Let $A = (a_{i,j})$ be a matrix. The *transpose* of A, denoted A^T, is its reflection through the diagonal; i.e. $A^T = (a_{j,i})$. If A is an $n \times n$ matrix, the *anti-transpose* of A is its reflection through the anti-diagonal; i.e. the matrix $(a_{n+1-i,n+1-j})$.

Definition 1.6.3. Let A and B be $n \times n$ matrices over the same field $\mathbb{F}$. Then:

- The *direct sum* of A and B, denoted $A \oplus B$, is the matrix

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

- The *anti-direct sum* of A and B, denoted $A \mathbin{\widehat{\oplus}} B$, is the matrix

$$\begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}.$$

Remark 1.6.4. Clearly, the direct sum of two diagonal matrices is diagonal, and the anti-direct sum of two anti-diagonal matrices is anti-diagonal.

The following lemma is immediate from the definitions:

Lemma 1.6.5. *Suppose A, B, C, D are square matrices of the same dimension over the same field. Then:*

$$(A \oplus B)(C \mathbin{\widehat{\oplus}} D) = BC \mathbin{\widehat{\oplus}} AD.$$
$$(A \mathbin{\widehat{\oplus}} B)(C \oplus D) = AC \mathbin{\widehat{\oplus}} BD.$$
$$(A \mathbin{\widehat{\oplus}} B)(C \mathbin{\widehat{\oplus}} D) = BC \oplus AD.$$

1.6.2 Forms

In order to define classical groups, we will first introduce the concept of a form.

Definition 1.6.6. Let $f = (f_{i,j}) \in \mathrm{M}_n(\mathbb{F})$. Then:

- f is *symmetric* if $f^T = f$.
- f is *anti-symmetric* if $f^T = -f$.
- f is *alternating* or *symplectic* if f is anti-symmetric and all of its diagonal entries are 0.
- f is *σ-hermitian* for some $\sigma \in \mathrm{Aut}(\mathbb{F})$ if $f^T = f^\sigma$, where $f^\sigma = (f^\sigma_{i,j})$.

We will often refer to f as a *symmetric/anti-symmetric/alternating/σ-hermitian form.* The first three are also said to be *bilinear forms.*

If $f \in \mathrm{GL}_n(\mathbb{F})$ (i.e. f is invertible) then f is said to be *non-degenerate*; otherwise, f is *degenerate*.

Definition 1.6.7. Let $\mathbb{F}$ be a field, f be a form on $\mathbb{F}^n$, and $g \in \mathrm{GL}_n(\mathbb{F})$. We say that g *preserves the form f up to scalars* if there exists $\lambda \in \mathbb{F}^*$ such that:

- If f is symmetric or anti-symmetric, then $gfg^T = \lambda f$;
- If f is σ-hermitian, then $gfg^{\sigma T} = \lambda f$.

If $\lambda = 1$, we say that g *preserves the form f*.

If $H < \mathrm{GL}_n(\mathbb{F})$, then H *preserves f up to scalars* (respectively H *preserves f*) provided every $h \in H$ preserves f up to scalars (respectively every $h \in H$ preserves f).

If $g \in \mathrm{GL}_n(\mathbb{F})$ preserves f up to scalars then we say that g is a *similarity* of f. If g preserves f then we say that g is an *isometry* of f.

Proposition 1.6.8. *Let $G < \mathrm{GL}_n(\mathbb{F})$ preserve a symmetric (respectively anti-symmetric) form f, and $c \in \mathrm{GL}_n(\mathbb{F})$. Then G^c preserves the symmetric (respectively anti-symmetric) form $c^{-1}fc^{-T}$.*

Proof. Let $g \in G$. Then:

$$\begin{aligned}(c^{-1}gc)(c^{-1}fc^{-T})(c^{-1}gc)^T &= (c^{-1}gc)(c^{-1}fc^{-T})(c^Tg^Tc^{-T})\\ &= c^{-1}gfg^Tc^{-T}\\ &= c^{-1}fc^{-T}\end{aligned}$$

so that $c^{-1}gc$ preserves $c^{-1}fc^{-T}$. Also, if $f^T = \pm f$, then $(c^{-1}fc^{-T})^T = c^{-1}f^Tc^{-T} = \pm c^{-1}fc^{-T}$, so that $c^{-1}fc^{-T}$ and f are bilinear forms of the same type. □

Proposition 1.6.9. *Let $G < \mathrm{GL}_n(\mathbb{F})$ preserve a σ-hermitian form f, and $c \in \mathrm{GL}_n(\mathbb{F})$. Then G^c preserves the σ-hermitian form $c^{-1}fc^{-\sigma T}$.*

Proof. Similar to Proposition 1.6.8. □

Definition 1.6.10. Let f and f' be forms over the same field $\mathbb{F}$, with σ a field automorphism of $\mathbb{F}$ if f is σ-hermitian and σ the trivial field automorphism otherwise. Then f and f' are *isometric* if there exists a matrix $c \in \mathrm{GL}_n(\mathbb{F})$ such that $f' = c^{-1}fc^{-\sigma T}$.

1.6.3 Extensions of classical groups

Remark 1.6.11. Typically, a finite classical group G will be the group of matrices in $\mathrm{GL}_n(q)$ that preserve a given form f. Together with G, we can associate a chain of groups $\Omega < S < G < C < \Gamma < A$, as described below:

- G is the *general group*, the group of matrices preserving a form f; in other words the group of isometries of f.
- S is the *special group* consisting of matrices preserving the form with determinant 1.
- Ω is equal to S except in the orthogonal case, and is usually quasisimple except in a few small cases. We will make both of these statements precise later.
- C is the *conformal group* consisting of matrices preserving the form up to scalars; in other words, the group of similarities of the form. In the natural representation this is the largest group in the chain that occurs as a subgroup of $\mathrm{GL}_n(q)$.
- Γ is the *conformal semilinear group*, which is usually the extension of C by field automorphisms of Ω.
- $A = \Gamma$ except in the linear case.

These groups will be defined precisely in the sections corresponding to the relevant classical groups. Often we will suppress mention of the form f, and in the following subsections we will describe the *standard form* preserved by each family of classical groups; whenever a classical group is introduced hereafter, we will assume that it preserves the standard form unless stated otherwise.

Throughout, adding the prefix P to a group name will denote the projective version of that group; that is, PX will denote the quotient of the group X by its centre $Z(X)$. This may also be denoted by $\overline{X}$.

Remark 1.6.12. We will not give a formal definition of what is meant by a classical group in this thesis, but will use the term in line with the usage in [8, p. x]. Thus, we will describe H as a classical group if we have $\Omega < H < A$ with Ω and A as in Remark 1.6.11 for any of the families of groups described in Sections 1.6.4-1.6.7. The term will also be used for the quotient of H by any subgroup of scalar matrices of H.

One automorphism that we will refer to frequently is the Frobenius automorphism.

Definition 1.6.13. Let $G = \mathrm{GL}_n(p^e)$. The *Frobenius automorphism* $\phi \in \mathrm{Aut}(G)$ is the automorphism mapping $(g_{i,j})$ to $(g_{i,j}^p)$. Likewise, for $r < e$ we define the *p^r-power Frobenius automorphism* to be ϕ^r.

Note that the Frobenius automorphism is obtained by applying the Frobenius mapping as defined in Definition 1.5.9 to each entry of $g \in G$.

In the following sections we will also construct the outer automorphism groups of each of the classical groups; the computations and notation throughout are due to [8, Section 1.7] unless cited otherwise. For each family of classical groups, we will also detail our notation for each of the groups Ω, S, G, C, Γ and A as given in Remark 1.6.11.

1.6.4 Linear groups

Construction

The general linear group $G = \mathrm{GL}_n(q)$ is the most straightforward example of a classical group. It is sometimes convenient to think of G as the group of invertible matrices preserving the degenerate bilinear form $f = 0$; viewed in this way, it is clear that the conformal linear group C is equal to the general linear group G.

G contains $S = \mathrm{SL}_n(q)$ as a subgroup.

Theorem 1.6.14. *[59, Section 3.3.2] The group* $\mathrm{PSL}_n(q) = \mathrm{SL}_n(q)/Z(\mathrm{SL}_n(q))$ *is simple unless* $(n, q) = (2, 2)$ *or* $(2, 3)$.

We denote the group $\mathrm{PSL}_n(q)$ by $\mathrm{L}_n(q)$.

Automorphisms

We now discuss the outer automorphism group of $\mathrm{L}_n(q)$. We give explicit generators for each generator of $\mathrm{Out}(\mathrm{L}_n(q))$, viewed as an element of $\mathrm{Aut}(\mathrm{L}_n(q))$. We will often abuse notation and use the same symbol for the element of $\mathrm{Out}(\mathrm{L}_n(q))$ and $\mathrm{Aut}(\mathrm{L}_n(q))$.

- δ is a *diagonal automorphism*, induced by $\mathrm{diag}(\nu, 1, \ldots, 1)$ where ν is a primitive element of $\mathbb{F}_q^*$. Projectively this extends $\mathrm{L}_n(q)$ to $\mathrm{PGL}_n(q)$.
- ϕ is a *field automorphism*, in this case the Frobenius automorphism as introduced in Definition 1.6.13. Projectively this extends $\mathrm{PGL}_n(q)$ to $\mathrm{P\Gamma L}_n(q)$.

- γ is a *graph automorphism*, sending g to g^{-T}. Projectively this extends $\mathrm{P\Gamma L}_n(q)$ to the full automorphism group $\mathrm{Aut}(\mathrm{L}_n(q))$. This is only outer for $n \geq 3$.

We also give a presentation of the outer automorphism group of $\mathrm{L}_n(q)$.

$$\mathrm{Out}(\mathrm{L}_2(q)) = \langle \delta, \phi \mid \delta^{(q-1,2)} = \phi^e = [\delta, \phi] = 1 \rangle.$$
$$\mathrm{Out}(\mathrm{L}_n(q)) = \langle \delta, \gamma, \phi \mid \delta^{(q-1,n)} = \gamma^2 = \phi^e = [\gamma, \phi] = 1, \delta^\gamma = \delta^{-1}, \delta^\phi = \delta^p \rangle \text{ for } n \geq 3.$$

1.6.5 Unitary groups

Construction

Let σ be the field automorphism of $\mathbb{F}_{p^{2e}}$ of order 2 (the q-power Frobenius automorphism), and let f be a non-degenerate σ-hermitian form over $\mathbb{F}_{q^2}$. Then the general unitary group $G = \mathrm{GU}_n(q, f) < \mathrm{GL}_n(q^2)$ is the group of matrices preserving f. Since σ is clear from the context, we will suppress mention of σ and instead typically refer to f as a *unitary form*.

Proposition 1.6.15. *[8, Proposition 1.5.28] Let f and f' be two non-degenerate unitary forms defined over $\mathbb{F}_{q^2}$. Then there exists a matrix $c \in \mathrm{GL}_n(q^2)$ such that $\mathrm{GU}_n(q, f)^c = \mathrm{GU}_n(q, f')$.*

Thus, any two non-degenerate unitary forms over $\mathbb{F}_{q^2}$ are isometric, and so often we will use the notation $\mathrm{GU}_n(q)$ in place of $\mathrm{GU}_n(q, f)$. If no form is explicitly specified, we will assume that the standard form preserved by $\mathrm{GU}_n(q)$ is I_n.

The special unitary group $S = \mathrm{SU}_n(q)$ is defined as in Remark 1.6.11, as matrices in $\mathrm{GU}_n(q)$ with determinant 1.

Theorem 1.6.16. *[59, Section 3.6.1] Suppose $n \geq 3$. If $(n, q) \neq (3, 2)$, then the group $\mathrm{PSU}_n(q) = \mathrm{SU}_n(q)/Z(\mathrm{SU}_n(q))$ is simple.*

We denote the group $\mathrm{PSU}_n(q)$ by $\mathrm{U}_n(q)$. When $n = 2$, we have that $\mathrm{U}_2(q) \cong \mathrm{L}_2(q)$.

The conformal unitary group $C = \mathrm{CGU}_n(q)$ and conformal semilinear unitary group $\Gamma = \Gamma\mathrm{U}_n(q)$ are also defined as in Remark 1.6.11.

Note that due to a result in [7, Section 3], some extensions of $\mathrm{SU}_n(q)$ and $\mathrm{U}_n(q)$ are not well-defined - specifically, their extensions by field automorphisms are not well-defined for even n and odd q, as there are two possible isomorphism classes depending on the form f. Thus we will describe the automorphisms of $\mathrm{U}_n(q)$ and $\mathrm{SU}_n(q)$ with respect to the form $f = I_n$ to avoid any ambiguity.

Remark 1.6.17. In many of the computations in the following sections, it is convenient to consider the linear and unitary groups together. For this reason, we will occasionally denote $\mathrm{U}_n(q)$ by $\mathrm{L}_n^-(q)$ (and the same for related groups), and use the notation $\mathrm{L}_n^{\pm}(q)$ to refer to a group which is either linear or unitary.

Automorphisms

We now discuss the outer automorphism group of $\mathrm{U}_n(q)$ preserving the form $f = I_n$. The group $\mathrm{Out}(\mathrm{U}_n(q))$ has the following generators:

- δ is a diagonal automorphism induced by $\mathrm{diag}(\nu^{q-1}, 1, \ldots, 1)$, where ν is a primitive element of $\mathbb{F}_{q^2}^*$. Projectively this extends $\mathrm{U}_n(q)$ to $\mathrm{PGU}_n(q)$.
- ϕ is the Frobenius automorphism as given in Definition 1.6.13. Projectively this extends $\mathrm{PGU}_n(q)$ to $\mathrm{P\Gamma U}_n(q)$.
- γ is a graph automorphism sending g to g^{-T}. Projectively this extends $\mathrm{P\Gamma U}_n(q)$ to $\mathrm{Aut}(\mathrm{U}_n(q))$.

We also give a presentation of the outer automorphism group.

$$\mathrm{Out}(\mathrm{U}_n(q)) = \langle \delta, \phi, \gamma \mid \delta^{(q+1,n)} = \gamma^2 = 1, \phi^e = \gamma, \delta^\gamma = \delta^{-1}, \delta^\phi = \delta^p \rangle \text{ for } n \geq 3.$$

1.6.6 Symplectic groups

Construction

Let $f \in \mathrm{GL}_n(q)$ be a non-degenerate symplectic form. Then the general symplectic group $G = \mathrm{Sp}_n(q, f) < \mathrm{GL}_n(q)$ is the group preserving f.

Proposition 1.6.18. *[8, Proposition 1.5.26] Let $f, f' \in \mathrm{GL}_n(q)$ be two nondegenerate symplectic forms. Then n is even, and there exists a matrix $c \in \mathrm{GL}_n(q)$ such that $\mathrm{Sp}_n(q, f)^c = \mathrm{Sp}_n(q, f')$.*

Thus any two non-degenerate symplectic forms over $\mathbb{F}_q$ are isometric. We will often use the notation $\mathrm{Sp}_n(q)$ in place of $\mathrm{Sp}_n(q, f)$. The standard symplectic form will be the matrix $\mathrm{antidiag}(-1, \ldots, -1, 1, \ldots, 1)$ with $\frac{n}{2}$ 1's and $\frac{n}{2}$ (-1)'s.

It follows from [53, Corollary 8.6] that the determinant of every element of $\mathrm{Sp}_n(q)$ is 1, and hence the special symplectic group is equal to the general symplectic group. We will refer to $\mathrm{Sp}_n(q)$ as the *symplectic group*.

Note that when $n = 2$, conjugation by the matrix $f = \mathrm{antidiag}(-1, 1)$ has the same action on $g \in \mathrm{SL}_2(q)$ as the inverse-transpose automorphism. (In particular, this means that the graph automorphism of $\mathrm{SL}_2(q)$ is not outer). This means that $\mathrm{Sp}_2(q) = \mathrm{SL}_2(q)$, and hence when we consider the symplectic group, we usually take $n \geq 4$.

Theorem 1.6.19. *[59, Section 3.5.2] Suppose $n \geq 4$ is even. If $(n, q) \neq (4, 2)$, then the group $\mathrm{PSp}_n(q) = \mathrm{Sp}_n(q)/Z(\mathrm{Sp}_n(q))$ is simple.*

We denote the group $\mathrm{PSp}_n(q)$ by $\mathrm{S}_n(q)$.

The conformal symplectic group $C = \mathrm{CSp}_n(q)$ and conformal semilinear symplectic group $\Gamma = \Gamma\mathrm{Sp}_n(q)$ are defined as in Remark 1.6.11. Unlike for the unitary groups, there is no ambiguity regarding the isomorphism type of $\Gamma\mathrm{Sp}_n(q)$; see [7, Theorem 4].

$\mathrm{P}\Gamma\mathrm{Sp}_n(q)$ is the full automorphism group of $\mathrm{S}_n(q)$, except when $n = 4$ and q is even where we have an exceptional graph automorphism.

Automorphisms

We now discuss the outer automorphism group of $\mathrm{S}_n(q)$ preserving the form $f = \mathrm{antidiag}(-1, \ldots, -1, 1, \ldots, 1)$. The group $\mathrm{Out}(\mathrm{S}_n(q))$ has the following generators:

- δ is a diagonal automorphism, which is nontrivial precisely when q is odd. It is induced by $\mathrm{diag}(\nu, \ldots, \nu, 1, \ldots, 1)$ with $\frac{n}{2}$ ν's and $\frac{n}{2}$ 1's, where ν is a primitive element of $\mathbb{F}_q$. Projectively, δ extends $\mathrm{S}_n(q)$ to $\mathrm{PCSp}_n(q)$.

- ϕ is the Frobenius automorphism as given in Definition 1.6.13. Projectively this extends $\mathrm{PCSp}_n(q)$ to $\mathrm{P}\Gamma\mathrm{Sp}_n(q)$.

- γ is a graph automorphism such that $\gamma^2 = \phi$, which only exists when $n = 4$ and $q = 2^{2r+1}$ for some integer r. This corresponds to the symmetry of the Dynkin diagram $\mathfrak{B}_2$ when both roots have the same length. We will not need an explicit definition of γ for this thesis; see for example [59, Section 4.2.1] for details.

We also give a presentation of the outer automorphism group.

$$\mathrm{Out}(\mathrm{S}_4(2^e)) = \langle \gamma, \phi | \gamma^2 = \phi, \phi^e = 1 \rangle \qquad \text{for } n \geq 4 \text{ even.}$$
$$\mathrm{Out}(\mathrm{S}_n(q)) = \langle \delta, \phi | \delta^{(q-1,2)} = \phi^e = [\delta, \phi] = 1 \rangle \qquad \text{for } (n, q) \neq (4, 2^e).$$

1.6.7 Orthogonal groups

Quadratic forms

Definition 1.6.20. The *upper triangulation* of an $n \times n$ matrix $g = (g_{i,j})$, denoted g^{UT}, is the matrix $(h_{i,j})$, where $h_{i,i} = g_{i,i}$ for $1 \leq i \leq n$, $h_{i,j} = g_{i,j} + g_{j,i}$ for $1 \leq i < j \leq n$ and $h_{i,j} = 0$ for $1 \leq j < i \leq n$.

A matrix Q is a *quadratic form* if it is upper triangular; i.e. if $Q = (q_{i,j})$ with $q_{i,j} = 0$ whenever $i > j$.

Let g be an $n \times n$ matrix, and Q an $n \times n$ quadratic form. Then g *preserves* Q if $(gQg^T)^{UT} = Q$.

The *orthogonal group* $\mathrm{GO}_n(q, Q)$ is the group of matrices preserving the quadratic form Q, defined over $\mathbb{F}_q$.

Remark 1.6.21.

(1) Associated with the matrix Q, we can obtain a symmetric bilinear form $f := Q + Q^T$. This f is called the *polar form* of Q.

(2) In fields of characteristic not equal to 2, we can recover Q from its polar form f by setting $Q = \frac{1}{2} f^{UT}$; thus in odd characteristic we have a one-to-one correspondence between symmetric bilinear forms and quadratic forms.

Proposition 1.6.22. *Let $G < \mathrm{GL}_n(\mathbb{F})$ preserve a quadratic form Q, and $c \in \mathrm{GL}_n(\mathbb{F})$. Then G^c preserves the quadratic form $(c^{-1}Qc^{-T})^{UT}$.*

Proof. Similar to Proposition 1.6.8. □

Definition 1.6.23. Let $G = \mathrm{GO}_n(q, Q)$ with associated polar form f, and let V be the underlying n-dimensional vector space. Suppose $v \in V$ is such that $vQv^T \neq 0$. Then we define the *reflection* $r_v : V \to V$ by

$$(x)r_v = x - \frac{vfx^T}{vQv^T} v = x - \frac{vQx^T + xQv^T}{vQv^T} v.$$

Remark 1.6.24. We have that $\det r_v = -1$ for any vector v such that $vQv^T \neq 0$.

Proposition 1.6.25. *[53, Corollary 11.42] Let $G = \mathrm{GO}_n(q, Q)$ with $(n, q) \neq (4, 2)$. Then G is generated by the set of reflections r_v.*

Remark 1.6.26. For some choices of Q, the group $G = \mathrm{GO}_4(2, Q)$ is also generated by the set of reflections. In the notation we will introduce later, if Q is a quadratic form of minus type then the set of reflections generates G; if Q is a quadratic form of plus type then G is not generated by reflections.

Definition 1.6.27. Suppose that $G = \mathrm{GO}_n(q, Q)$ with G generated by reflections, and $g \in G$. Write $g = \prod_{i=1}^{k} r_{v_i}$, where v_i are vectors.

- If q is odd, then the *spinor norm* of g is $+1$ if $\prod_{i=1}^{k} v_i f v_i^T$ is a square in $\mathbb{F}_q^*$. Otherwise, the spinor norm is -1.
- If q is even, then the *quasideterminant* of g is $+1$ if k is even, and -1 if k is odd.

Proposition 1.6.28. *[59, Section 3.8.1 and Section 3.9.2] The spinor norm and the quasideterminant are well-defined homomorphisms* $\mathrm{GO}_n(q, Q) \to \{\pm 1\}$.

Remark 1.6.29.

(1) Observe that our definition of the spinor norm homomorphism has image ± 1, in comparison with the determinant homomorphism. However elsewhere, including Magma, the convention is to use 0 and 1 in place of 1 and -1 respectively, which may cause some confusion in some of the Magma computations later in this thesis.

(2) When q is odd, the kernels K_1 of the determinant homomorphism and K_2 of the spinor norm homomorphism are distinct subgroups of $\mathrm{GO}_n(q, Q)$ of index two. There are alternative methods of defining the spinor norm homomorphism, which all agree on elements of $K_1 \cap K_2$ (matrices with determinant and spinor norm 1); however some definitions may take the kernel of the spinor norm to be $K_1 K_2$ instead of K_2. Hence, there is not a universally well-defined notion of a matrix with determinant 1 and spinor norm -1 with respect to a form f, and in particular there may be computational variations when considering the spinor norms of elements with determinant -1.

The following lemma provides a method of computing the spinor norm.

Lemma 1.6.30. *[8, Proposition 1.6.11] Let $g \in \mathrm{GO}_n(q, Q)$ with associated polar form f. Let $a := I_n - g$ and suppose that a has rank k. If q is odd, then let b be a $k \times n$ matrix over $\mathbb{F}_q$ whose rows form a basis of a complement of the nullspace of a. Then:*

(i) If q is even and $\mathrm{GO}_n(q, Q)$ is generated by reflections, then the quasideterminant of g is 1 if k is even and -1 if k is odd.

(ii) If q is odd, then the spinor norm of g is 1 if $\det(bafb^T)$ is a square in $\mathbb{F}_q^*$, and -1 otherwise.

We will now consider separately the cases when the field has odd characteristic and even characteristic.

Orthogonal groups in odd characteristic

In odd characteristic, the following proposition shows that it suffices to consider symmetric bilinear forms rather than quadratic forms.

Proposition 1.6.31. *[8, Proposition 1.5.15] Let $\mathbb{F}$ be a field with $\mathrm{char}(\mathbb{F}) \neq 2$. Let Q be a quadratic form over $\mathbb{F}$, with associated polar form f. Then a matrix g preserves f if and only if it preserves Q.*

Thus, we write $\mathrm{GO}_n(q, f)$ instead of $\mathrm{GO}_n(q, Q)$, where f is the polar form of Q. It follows immediately from the fact that $gfg^T = f$ that $\det g = \pm 1$ for any $g \in \mathrm{GO}_n(q, f)$.

Proposition 1.6.32. *[53, p. 138-139] Let f and f' be $n \times n$ symmetric bilinear forms over $\mathbb{F}_q$, where q is odd.*

(i) If n is even, then there exists a matrix $c \in \mathrm{GL}_n(q)$ such that $\mathrm{GO}_n(q, f)^c = \mathrm{GO}_n(q, f')$ if and only if $\det(ff')$ is a square.

(ii) If n is odd, then there exists a matrix $c \in \mathrm{GL}_n(q)$ such that $\mathrm{GO}_n(q, f)^c = \mathrm{GO}_n(q, f')$.

Remark 1.6.33. It follows immediately from Proposition 1.6.32(ii) that when n is odd, the isomorphism type of the group $\mathrm{GO}_n(q, f)$ does not depend on f. We will denote this group by $\mathrm{GO}_n^\circ(q, f)$, and often we will simply refer to it as $\mathrm{GO}_n^\circ(q)$. We will take the standard form to be I_n in this case.

By Proposition 1.6.32(i), when n is even we have two isomorphism classes of groups preserving a symmetric bilinear form. We denote these two groups by $\mathrm{GO}_n^+(q)$ and $\mathrm{GO}_n^-(q)$. The standard forms preserved by $\mathrm{GO}_n^\pm(q)$ are as follows:

- The standard form for $\mathrm{GO}_n^+(q)$ is $\mathrm{antidiag}(1, \ldots, 1)$.
- Let $f_t = \mathrm{diag}(t, 1, \ldots, 1)$ for some choice of t, and let ν denote a primitive element of $\mathbb{F}_q$. Note that by Proposition 1.6.32(i), the groups of matrices preserving f_1 and f_ν are not isometric; hence one will be isometric to $\mathrm{antidiag}(1, \ldots, 1)$ and we will take the other to be the standard form preserved by $\mathrm{GO}_n^-(q)$. When

$n \equiv 2 \mod 4$ and $q \equiv 3 \mod 4$, I_n is not isometric to antidiag$(1, \ldots, 1)$; hence in this case the standard form preserved by $\mathrm{GO}_n^-(q)$ is I_n. Otherwise I_n and antidiag$(1, \ldots, 1)$ are isometric, and the standard form preserved by $\mathrm{GO}_n^-(q)$ is diag$(\nu, 1, \ldots, 1)$.

We will thus refer to a generic orthogonal group as $\mathrm{GO}_n^\epsilon(q)$, where $\epsilon = \circ$ if n is odd, and $\epsilon \in \{+, -\}$ if n is even. We will refer to these groups as the *orthogonal group*, *orthogonal plus-type group* and *orthogonal minus-type group* respectively.

We define $S = \mathrm{SO}_n^\epsilon(q)$ to be the subgroup of $\mathrm{GO}_n^\epsilon(q)$ of matrices with determinant 1. Unlike the other classical groups, $\mathrm{SO}_n^\epsilon(q)$ is not quasisimple; indeed it contains a (unique) index-2 subgroup denoted $\Omega = \Omega_n^\epsilon(q)$, the kernel of the restriction of the spinor norm homomorphism as defined in Definition 1.6.27.

Theorem 1.6.34. *[59, Section 3.7.3] Suppose $n \geq 5$ and q is odd. Then the group $\mathrm{P}\Omega_n^\epsilon(q) = \Omega_n^\epsilon(q)/Z(\Omega_n^\epsilon(q))$ is simple.*

We denote the group $\mathrm{P}\Omega_n^\epsilon(q)$ by $\mathrm{O}_n^\epsilon(q)$.

Remark 1.6.35.

(1) It follows from [53, Theorem 11.4] that $\mathrm{GO}_2^+(q)$ and $\mathrm{GO}_2^-(q)$ are dihedral groups. The corresponding subgroups $\Omega_2^\pm(q)$ are cyclic.

(2) It is shown in [53, Theorem 11.6] that $\Omega_3^\circ(q) \cong \mathrm{L}_2(q)$.

(3) We also have the following isomorphisms, with sketch proofs in [33, Proposition 2.9.1]:

  (i) $\Omega_4^+(q) \cong (2, q-1)^{\cdot}(\mathrm{L}_2(q) \times \mathrm{L}_2(q))$.

  (ii) $\Omega_4^-(q) \cong \mathrm{L}_2(q^2)$.

  (iii) $\Omega_5^\circ(q) \cong \mathrm{S}_4(q)$.

  (iv) $\mathrm{O}_6^+(q) \cong \mathrm{L}_4(q)$.

  (v) $\mathrm{O}_6^-(q) \cong \mathrm{U}_4(q)$.

Following Remark 1.6.35 we will only consider the orthogonal groups in dimension $n \geq 7$.

The conformal orthogonal groups $C = \mathrm{CGO}_n^\epsilon(q)$ and conformal semilinear orthogonal groups $\Gamma = \mathrm{C\Gamma O}_n^\epsilon(q)$ are defined as in Remark 1.6.11. The group $\mathrm{PC\Gamma O}_n^\epsilon(q)$ is the full automorphism group of $\mathrm{O}_n^\epsilon(q)$ except when $n = 8$ and $\epsilon = +$, where there is an exceptional graph automorphism of order 3.

When n is even and q is a square, it follows from [7, Proposition 12] that there are two isomorphism classes of field extensions of $\Omega_n^{\epsilon}(q)$. In particular, the automorphism groups we describe below will be with respect to the standard forms as given in Remark 1.6.33.

Automorphisms in odd characteristic

We now discuss the outer automorphism group of $\mathrm{O}_n^{\epsilon}(q)$ for q odd and forms as in Remark 1.6.33. For n even, the group $\mathrm{Out}(\mathrm{O}_n^{+}(q))$ has the following generators:

- δ' is a diagonal automorphism which is nontrivial unless $n \equiv 2 \mod 4$ and $q \equiv 3 \mod 4$. It is induced by an element of $\mathrm{SO}_n^{+}(q) \setminus \Omega_n^{+}(q)$. Projectively, δ' extends $\mathrm{O}_n^{+}(q)$ to $\mathrm{PSO}_n^{+}(q)$.
- γ is a graph automorphism. It is induced by an element of $\mathrm{GO}_n^{+}(q) \setminus \mathrm{SO}_n^{+}(q)$. Projectively, γ extends $\mathrm{PSO}_n^{+}(q)$ to $\mathrm{PGO}_n^{+}(q)$.
- δ is a diagonal automorphism. It is induced by conjugation by the matrix $\mathrm{diag}(\nu, \dots, \nu, 1, \dots, 1)$ in $\mathrm{CGO}_n^{+}(q) \setminus \mathrm{GO}_n^{+}(q)$. This matrix scales the form by ν and has determinant $\nu^{\frac{n}{2}}$. Projectively it extends $\mathrm{PGO}_n^{+}(q)$ to $\mathrm{PCGO}_n^{+}(q)$.
- ϕ is the Frobenius automorphism as given in Definition 1.6.13. Projectively this extends $\mathrm{PCGO}_n^{+}(q)$ to $\mathrm{PC\Gamma O}_n^{+}(q)$.
- τ is a graph automorphism of order 3, which is only defined when $n = 4$. This corresponds to the fact that the Dynkin diagram $\mathfrak{D}_4$ has additional symmetry. We will not provide an explicit definition of τ here; see for example [59, Section 4.7] for details.

We also give a presentation of the outer automorphism group.

Conditions	$\mathrm{Out}(\mathrm{O}_n^{+}(q))$
$n \geq 12, 4 \mid n$	$\langle \delta', \gamma, \delta, \phi \mid \delta'^2 = \gamma^2 = \delta^2 = 1, (\delta\gamma)^2 = \delta', \phi^e = [\delta, \phi] = [\gamma, \phi] = 1 \rangle$
$n \geq 6, n \equiv 2 \mod 4, q \equiv 1 \mod 4$	$\langle \delta', \gamma, \delta, \phi \mid \delta'^2 = \gamma^2 = 1, \delta^2 = \delta', \delta^{\gamma} = \delta^{-1}, \phi^e = [\gamma, \phi] = 1, \delta^{\phi} = \delta^p \rangle$
$n \geq 6, n \equiv 2 \mod 4, q \equiv 3 \mod 4$	$\langle \gamma, \delta, \phi \mid \gamma^2 = \delta^2 = [\delta, \gamma] = \phi^e = [\gamma, \phi] = [\delta, \phi] = 1 \rangle$
$n = 8$	$\langle \delta', \tau, \gamma, \delta, \phi \mid \delta'^2 = \tau^3 = \gamma^2 = (\gamma\tau)^2 = \delta^2 = 1, \delta^{\tau} = \delta',$ $\delta'^{\tau} = \delta\delta', (\delta\gamma)^2 = \delta', \phi^e = [\delta, \phi] = [\tau, \phi] = [\gamma, \phi] = 1 \rangle$

For n even, the group $\mathrm{Out}(\mathrm{O}_n^-(q))$ has the following generators:

- δ' is a diagonal automorphism which is only defined when $n \equiv 2 \mod 4$ and $q \equiv 3 \mod 4$. It is induced by an element of $\mathrm{SO}_n^-(q) \setminus \Omega_n^-(q)$. Projectively, δ' extends $\mathrm{O}_n^-(q)$ to $\mathrm{PSO}_n^-(q)$.

- γ is a graph automorphism. It is induced by an element of $\mathrm{GO}_n^-(q) \setminus \mathrm{SO}_n^-(q)$. Projectively, γ extends $\mathrm{PSO}_n^-(q)$ to $\mathrm{PGO}_n^-(q)$.

- δ is a diagonal automorphism, induced by an element of $\mathrm{CGO}_n^-(q) \setminus \mathrm{GO}_n^-(q)$, which is defined as follows. Choose $a, b \in \mathbb{F}_q^*$ such that $a^2 + b^2 = \nu$, for ν a primitive element of $\mathbb{F}_q^*$. Define $X = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & \nu \\ -1 & 0 \end{pmatrix}$. Then $\delta = \mathrm{diag}(X, \ldots, X)$ when $n \equiv 2 \mod 4$ and $q \equiv 3 \mod 4$, and $\delta = \mathrm{diag}(Y, X, \ldots, X)$ otherwise. In both cases, this matrix scales the form by ν and has determinant $\nu^{\frac{n}{2}}$. Projectively this extends $\mathrm{PGO}_n^-(q)$ to $\mathrm{PCGO}_n^-(q)$.

- ϕ is the Frobenius automorphism as given in Definition 1.6.13, and is only defined when $n \equiv 2 \mod 4$ and $q \equiv 3 \mod 4$. Projectively this extends $\mathrm{PCGO}_n^-(q)$ to $\mathrm{PC\Gamma O}_n^-(q)$, which is the full automorphism group.

- φ is the field automorphism when $4|n$ or $q \equiv 1 \mod 4$, and is given by ϕc_g, where ϕ is the Frobenius automorphism as given in Definition 1.6.13 and c_g is conjugation by the matrix $g = \mathrm{diag}(\nu^{(p-1)/2}, 1, 1, \ldots, 1)$. Projectively this extends $\mathrm{PCGO}_n^-(q)$ to $\mathrm{PC\Gamma O}_n^-(q)$, which is the full automorphism group.

We also give a presentation of the outer automorphism group.

Conditions	$\mathrm{Out}(\mathrm{O}_n^-(q))$
$n \geq 4, 4\|n$ or $q \equiv 1 \mod 4$	$\langle \gamma, \delta, \varphi \| \gamma^2 = \delta^2 = [\delta, \gamma] = [\delta, \varphi] = 1, \varphi^e = \gamma \rangle$
$n \geq 4, n \equiv 2 \mod 4$ and $q \equiv 3 \mod 4$	$\langle \delta', \gamma, \delta, \phi \| \delta'^2 = \gamma^2 = 1, \delta^2 = \delta', \delta^\gamma = \delta^{-1}, \phi^e = [\gamma, \phi] = [\delta, \phi] = 1 \rangle$

For n odd, the group $\mathrm{Out}(\mathrm{O}_n^\circ(q))$ has the following generators:

- δ is a diagonal automorphism, induced by an element of $\mathrm{SO}_n^\circ(q) \setminus \Omega_n^\circ(q)$. Projectively, this extends $\mathrm{O}_n^\circ(q)$ to $\mathrm{PSO}_n^\circ(q)$.

- ϕ is the Frobenius automorphism. Projectively, this extends $\mathrm{PSO}_n^\circ(q)$ to $\mathrm{PC\Gamma O}_n^\circ(q)$, which is the full automorphism group.

We also give a presentation of the outer automorphism group.

$$\mathrm{Out}(\mathrm{O}_n^\circ(q)) = \langle \delta, \phi | \delta^2 = \phi^e = [\delta, \phi] = 1 \rangle.$$

Orthogonal groups in even characteristic

Throughout this section let q be even. We first describe the situation when n is odd.

Lemma 1.6.36. *[53, Theorem 11.9] Let q be even and n be odd. Then the natural module of $\mathrm{O}_n^\circ(q)$ is reducible, and $\mathrm{O}_n^\circ(q) \cong \mathrm{S}_{n-1}(q)$.*

Let us now consider the case when n is even.

Remark 1.6.37. When n is even, there are two isomorphism classes of groups $\mathrm{GO}_n(q, Q)$, depending on the Witt index of Q. We will not define the Witt index here; see for instance [53, p. 59]. It will suffice for our purposes to describe our standard quadratic forms.

As in the case when q is odd, we will denote the two isomorphism classes $\mathrm{GO}_n^+(q)$ and $\mathrm{GO}_n^-(q)$. The standard quadratic form for $\mathrm{GO}_n^+(q)$ is

$$\mathrm{antidiag}(0, \ldots, 0, 1, \ldots, 1).$$

The standard quadratic form for $\mathrm{GO}_n^-(q)$ is

$$\mathrm{antidiag}(0, \ldots, 0, 1, \ldots, 1) + E_{\frac{n}{2}, \frac{n}{2}} + \mu E_{\frac{n}{2}+1, \frac{n}{2}+1},$$

where $\mu \in \mathbb{F}_q$ is such that the polynomial $x^2 + x + \mu$ is irreducible over $\mathbb{F}_q$.

Note that in even characteristic, all matrices preserving a quadratic form have determinant 1, so that $S = \mathrm{SO}_n^\pm(q) = \mathrm{GO}_n^\pm(q)$. We define $\Omega = \Omega_n^\pm(q)$ as the kernel of the quasideterminant homomorphism.

Theorem 1.6.38. *[59, Section 3.8.2] Suppose $n \geq 6$ is even, and q is even. Then the group $\mathrm{P}\Omega_n^\pm(q) = \Omega_n^\pm(q)/Z(\Omega_n^\pm(q))$ is simple.*

We denote the group $\mathrm{P}\Omega_n^\pm(q)$ by $\mathrm{O}_n^\pm(q)$.

The conformal orthogonal group $C = \mathrm{CGO}_n^\pm(q)$ is equal to the general orthogonal group $\mathrm{GO}_n^\pm(q)$, and we define the conformal semilinear orthogonal group $\mathrm{C\Gamma O}_n^\pm(q)$ in the usual way.

As in the case where q is odd, to avoid ambiguity with respect to extensions of $\Omega_n^\epsilon(q)$ by field automorphism, we will describe the automorphisms in the following section with respect to the standard forms.

Automorphisms in even characteristic

We now discuss the outer automorphism group of $\mathrm{O}_n^\epsilon(q)$ for q even, n even and forms as in Remark 1.6.37. The group $\mathrm{Out}(\mathrm{O}_n^\pm(q))$ has the following generators:

- γ is a graph automorphism induced by an element of $\mathrm{SO}_n^\pm(q) \setminus \Omega_n^\pm(q)$. Projectively this extends $\mathrm{O}_n^\pm(q)$ to $\mathrm{PGO}_n^\pm(q)$.
- ϕ is the Frobenius automorphism as given in Definition 1.6.13, when the group is $\Omega_n^+(q)$. Projectively this extends $\mathrm{PGO}_n^+(q)$ to $\mathrm{P C \Gamma O}_n^+(q)$, which is the full automorphism group except when $n = 4$.
- φ is the field automorphism when the group is $\Omega_n^-(q)$, and is given by ϕc_g, where ϕ is the Frobenius automorphism as given in Definition 1.6.13 and c_g is conjugation by a matrix g which we will not define explicitly here; see [33, Section 2.8] for details. Projectively this extends $\mathrm{PGO}_n^-(q)$ to $\mathrm{P C \Gamma O}_n^-(q)$, which is the full automorphism group.
- τ is a graph automorphism of order 3 when $n = 4$, as in the odd characteristic case.

We also give a presentation of the outer automorphism groups.

Conditions	$\mathrm{Out}(\mathrm{O}_n^\pm(q))$
$n = 8, \epsilon = +$	$\langle \tau, \gamma, \phi \mid \tau^3 = \gamma^2 = (\gamma\tau)^2 = \phi^e = [\tau, \phi] = [\gamma, \phi] = 1 \rangle$
$n \neq 8, \epsilon = +$	$\langle \gamma, \phi \mid \gamma^2 = \phi^e = [\gamma, \phi] = 1 \rangle$
$\epsilon = -$	$\langle \gamma, \varphi \mid \gamma^2 = 1, \varphi^e = \gamma \rangle$

1.7 Representation theory

In this section we provide a brief introduction to the aspects of representation theory that we will need for this thesis. For a more detailed treatment of representation and character theory, see [30].

1.7.1 Representations

Definition 1.7.1. A *representation* of a group G is a homomorphism $\rho : G \to \mathrm{GL}_n(\mathbb{F})$ for some integer n and field $\mathbb{F}$. The integer n is called the *dimension* or *degree* of the representation, and if we want to specify $\mathbb{F}$ precisely we will say that ρ is a representation of G over $\mathbb{F}$. The image $G\rho$ is said to be the *action group* of the representation ρ.

A *projective representation* of G is a homomorphism $\rho : G \to \mathrm{PGL}_n(\mathbb{F})$.

For our purposes, $\mathbb{F}$ will usually be either a finite field, a number field (see Section 1.9.2) or $\mathbb{C}$.

It is a straightforward result in representation theory (see for instance [30, Chapter 1]) that there is an equivalence between representations ρ of degree n and G-modules $V = \mathbb{F}^n$, where G acts on V by $v \cdot g = v(g\rho)$. Thus we will often refer to the G-module V as also being the representation of G, and will interchange the two notions without further comment.

Definition 1.7.2. Let $\rho : G \to \mathrm{GL}_n(\mathbb{F})$ be a representation.

(i) The representation ρ is *faithful* if $G\rho \cong G$; i.e. ρ is injective.

(ii) The representation ρ is *irreducible* if $G\rho$ stabilises no proper non-zero subspace of $\mathbb{F}^n$; otherwise ρ is *reducible.*

(iii) The representation ρ is *absolutely irreducible* if, for any field $\mathbb{E} > \mathbb{F}$, we have that ρ is irreducible, where $G\rho$ is viewed as a subgroup of $\mathrm{GL}_n(\mathbb{E})$.

Lemma 1.7.3. *[30, Theorem 9.2] A representation $\rho : G \to \mathrm{GL}_n(\mathbb{F})$ is absolutely irreducible if and only if the only matrices in $\mathrm{GL}_n(\mathbb{F})$ which centralise $G\rho$ are scalar matrices.*

Remark 1.7.4. Note that if we have a representation $\rho : G \to \mathrm{GL}_n(\mathbb{F})$, and a subgroup $H < G$, then the restriction of ρ to H gives us a representation of H, which we denote $\rho|_H$. If ρ is reducible then $\rho|_H$ will also be reducible, but it is possible for ρ to be irreducible (respectively absolutely irreducible) and $\rho|_H$ to be reducible (respectively not absolutely irreducible).

We now introduce some notation. Let $\rho : G \to \Omega$ be a representation, for some classical group Ω. Let $\alpha \in \mathrm{Aut}(G)$ and $\beta \in \mathrm{Aut}(\Omega)$. We denote by ${}^{\alpha}\rho$ the representation given by $g({}^{\alpha}\rho) = (g^{\alpha})\rho$, and by ρ^{β} the representation given by $g(\rho^{\beta}) = (g\rho)^{\beta}$.

Definition 1.7.5. Let ρ and ρ' be representations $G \to \mathrm{GL}_n(\mathbb{F})$.

(i) The representations ρ and ρ' are *equivalent* if there exists $\beta \in \mathrm{Inn}(\mathrm{GL}_n(\mathbb{F}))$ such that $\rho' = \rho^{\beta}$; i.e. there exists a matrix $x \in \mathrm{GL}_n(\mathbb{F})$ such that $x^{-1}(g\rho)x = g\rho'$ for every $g \in G$.

(ii) The representations ρ and ρ' are *quasi-equivalent* if there exists $\alpha \in \mathrm{Aut}(G)$ such that ${}^{\alpha}\rho$ is equivalent to ρ.

(iii) The automorphism $\alpha \in \mathrm{Aut}(G)$ *stabilises* ρ if ${}^{\alpha}\rho$ is equivalent to ρ.

(iv) The automorphism $\beta \in \mathrm{Aut}(\mathrm{GL}_n(\mathbb{F}))$ *stabilises* ρ if ρ^{β} is equivalent to ρ.

1.7.2 Representations preserving forms

We first make the following definition of the dual representation, following [8, Proposition 1.8.3].

Definition 1.7.6. Let $\rho : G \to \mathrm{GL}_n(\mathbb{F})$ be a representation. Then we define the *dual representation* of ρ, denoted ρ^*, by $g\rho^* = (g\rho)^{-T}$.

The representation ρ is *self-dual* if ρ is equivalent to ρ^*.

Next, we show some connections between duality of absolutely irreducible representations ρ and the type of form preserved by $G\rho$.

Lemma 1.7.7. *Let $\rho : G \to \mathrm{GL}_n(\mathbb{F})$ be an absolutely irreducible self-dual representation of G. Then $G\rho$ preserves a nondegenerate symmetric or anti-symmetric bilinear form.*

Proof. Let V denote the module corresponding to ρ. Since ρ is self-dual, it follows that ρ and ρ^* are equivalent, so there exists a matrix $x \in \mathrm{GL}_n(\mathbb{F})$ such that for all $g \in G$, $x^{-1}(g\rho)x = g\rho^* = (g\rho)^{-T}$. Rearranging, we get that $x = (g\rho)x(g\rho)^T$; hence if x is symmetric or anti-symmetric then $G\rho$ preserves the corresponding bilinear form. Transposing this relation, we also get that $x^T = (g\rho)x^T(g\rho)^T$, and thus also that $x^{-T}(g\rho)x^T = (g\rho)^{-T}$. Hence for all $g \in G$ we have

$$xx^{-T}(g\rho)x^Tx^{-1} = x(g\rho)^{-T}x^{-1} = g\rho.$$

Hence x^Tx^{-1} centralises $G\rho$, and since $G\rho$ is absolutely irreducible it follows from Lemma 1.7.3 that there exists $\lambda \in \mathbb{F}^*$ such that $x = \lambda x^T$. Transposing, we obtain that $x^T = \lambda x$ so that $x = \lambda^2 x$ and $\lambda = \pm 1$. Thus either $x = x^T$ or $x = -x^T$; i.e. x is either a symmetric or anti-symmetric matrix. $\square$

Lemma 1.7.8. *Let $\rho : G \to \mathrm{GL}_n(\mathbb{F})$ be an absolutely irreducible representation,. Let σ be a field automorphism of $\mathbb{F}$ of order 2, and suppose that $(\rho^*)^\sigma$ is equivalent to ρ. Then $G\rho$ preserves a σ-sesquilinear form.*

Proof. Similar to Lemma 1.7.7. $\square$

Remark 1.7.9. Conversely, if $G\rho$ preserves a nondegenerate symmetric or anti-symmetric bilinear form, then ρ is self-dual (although not necessarily absolutely irreducible). Similarly, if $G\rho$ preserves a σ-sesquilinear form, then ρ is equivalent to $(\rho^*)^\sigma$.

Lemma 1.7.10. *[8, Lemma 1.8.8] Let $\rho : G \to \mathrm{GL}_n(\mathbb{F})$ be an absolutely irreducible representation of G.*

(i) Up to multiplication of the form by a scalar, $G\rho$ preserves at most one bilinear form, at most one σ-hermitian form for a given automorphism σ of $\mathbb{F}$, and (assuming $\mathrm{char}(\mathbb{F}) \neq 2$ or $\mathbb{F}$ is finite) at most one quadratic form.

(ii) If $\mathbb{F}$ is finite and $G\rho$ simultaneously preserves a σ-hermitian form and a bilinear form, then $G\rho$ is conjugate to a subgroup of $\mathrm{GL}_n(\mathbb{E})$ for some proper subfield $\mathbb{E}$ of $\mathbb{F}$.

We next collect some results on matrices inducing automorphisms on quasi-equivalent representations.

Lemma 1.7.11. *[8, Lemma 1.8.9] Let $\rho : G \to \mathrm{GL}_n(\mathbb{F})$ be an absolutely irreducible representation, and suppose that $G\rho$ lies inside the general group of a classical group preserving a form f. If the form preserved is a quadratic form and $\mathbb{F}$ has characteristic 2, assume additionally that $\mathbb{F}$ is finite. Let C denote the group of similarities of the form f. Then $N_{\mathrm{GL}_n(\mathbb{F})}(G\rho) \leq C$.*

Lemma 1.7.12. *[8, Lemma 1.8.6] Let $\rho, \rho' : G \to \mathrm{GL}_n(\mathbb{F})$ be faithful, and let $\alpha \in \mathrm{Aut}(G)$. Then:*

(i) Representations ρ and ρ' are quasi-equivalent if and only if $G\rho$ and $G\rho'$ are conjugate in $\mathrm{GL}_n(\mathbb{F})$.

(ii) The automorphism α stabilises ρ if and only if there exists $x \in \mathrm{GL}_n(\mathbb{F})$ which normalises $G\rho$ and induces α on $G\rho$.

Lemma 1.7.13. *[8, Lemma 1.8.10] Let $\rho, \rho' : G \to \mathrm{GL}_n(q)$ be quasi-equivalent absolutely irreducible representations of a group G. Then:*

(i) If $G\rho$ and $G\rho'$ are both subgroups of $\mathrm{GU}_n(\sqrt{q})$, then $G\rho$ and $G\rho'$ are conjugate in $\mathrm{GU}_n(\sqrt{q})$.

(ii) If $G\rho$ and $G\rho'$ are both subgroups of $\mathrm{Sp}_n(q)$, then $G\rho$ and $G\rho'$ are conjugate in $\mathrm{CSp}_n(q)$.

(iii) If $G\rho$ and $G\rho'$ are both subgroups of $\mathrm{GO}_n^\epsilon(q)$, then $G\rho$ and $G\rho'$ are conjugate in $\mathrm{CGO}_n^\epsilon(q)$.

1.7.3 Character theory

An object closely related to a representation of a group G is a character of the group.

Definition 1.7.14. The *character* of a representation ρ of a group G is the function $\chi : G \to \mathbb{F}$ gven by $g\chi = \mathrm{Trace}(g\rho)$ for all $g \in G$. The *character ring* of a character is the smallest subring of $\mathbb{C}$ containing the image $G\chi$. Similarly, the *character field* is the smallest subfield of $\mathbb{C}$ containing the character values.

There is an important connection between characters and representations.

Lemma 1.7.15. *[30, Corollary 2.9] Let $\rho, \rho' : G \to \mathrm{GL}_n(\mathbb{C})$ be representations of a group G, with corresponding characters χ and χ' respectively. Then $\chi = \chi'$ if and only if ρ and ρ' are equivalent.*

Since many of the representations we will be considering in this thesis will be obtained from representations over $\mathbb{C}$, this connection between representations and characters will be used frequently. In particular, we will refer to a character being irreducible, absolutely irreducible, self-dual etc. if the corresponding representation is.

When considering containments later, we will also need the notion of an induced character.

Definition 1.7.16. Let G be a group, $H < G$ and χ a character of H. Then the *induced character* on G is given by:

$$g\chi^G = \frac{1}{|H|} \sum_{x \in G} (x^{-1}gx)\chi^\circ$$

where $h\chi^\circ = \begin{cases} h\chi & \text{if } h \in H, \\ 0 & \text{if } h \notin H. \end{cases}$

Lemma 1.7.17. *[30, Corollary 5.3] The induced character χ^G is a character of G.*

Representation and character theory can vary significantly if we change the field of definition of the representation, depending on whether the characteristic of the field divides the order of the group G. Characters over fields of characteristic dividing the order of G are called *Brauer characters.* We will not require much of the theory of Brauer characters in this thesis - see for instance [30, Chapter 15] for a full introduction to the theory.

1.7.4 Other results

The following results will also be useful in later sections.

Lemma 1.7.18. *[15, Theorem 29.7] and [8, Proposition 1.8.12]. Let $\mathbb{F} < \mathbb{E}$ be fields, and $\rho, \rho' : G \to \mathrm{GL}_n(\mathbb{F})$ be representations. Then if ρ and ρ' are equivalent as representations over $\mathbb{E}$, then they are equivalent over $\mathbb{F}$.*

Lemma 1.7.19. *[14, Theorem 74.9] and [8, Proposition 1.8.13]. Let $\rho : G \to \mathrm{GL}_n(\mathbb{E})$ be an absolutely irreducible representation of G with corresponding character χ, with $\mathrm{char}(\mathbb{E}) = p > 0$. Let $\mathbb{F}$ denote the smallest subfield of $\mathbb{E}$ containing the base field $\mathbb{F}_p$ and the character values $g\chi$ for all $g \in G$. Then ρ is equivalent over $\mathbb{E}$ to a representation with image in $\mathrm{GL}_n(\mathbb{F})$.*

Corollary 1.7.20. *[8, Corollary 1.8.14] Let ρ and $\mathbb{E}$ be as in Lemma 1.7.19. If ρ is equivalent to ρ^ϕ for an automorphism ϕ of $\mathbb{E}$, then ρ is equivalent to a representation with image in $\mathrm{GL}_n(\mathbb{K})$, where $\mathbb{K}$ is the fixed field of ϕ.*

1.8 Tensor products

1.8.1 Definitions

We provide a very brief introduction to tensor products of representations and matrices.

Definition 1.8.1. Let V and W be vector spaces over a field $\mathbb{F}$ with bases $\{v_i : i \in I\}$ and $\{w_j : j \in J\}$ respectively. Then the *tensor product* of V and W, denoted $V \otimes W$, is a vector space with basis $\{v_i \otimes w_j : i \in I, j \in J\}$ satisfying the following relations, for $a, b \in I$, $c, d \in J$, $\lambda \in \mathbb{F}$:

- $(v_a + v_b) \otimes w_c = (v_a \otimes w_c) + (v_b \otimes w_c)$.
- $v_a \otimes (w_c + w_d) = (v_a \otimes w_c) + (v_a \otimes w_d)$.
- $\lambda(v_a \otimes w_c) = (\lambda v_a) \otimes w_c = v_a \otimes (\lambda w_c)$.

Remark 1.8.2. This construction can be extended in a straightforward way when V and W are representations of a group G to give a representation $V \otimes W$ of G; see [30, Chapter 4] for details.

Definition 1.8.3. Let $\mathbb{F}$ be a field, and $A = (a_{i,j})_{a \times b}$ and $B = (b_{k,l})_{c \times d}$, with $a_{i,j}, b_{k,l} \in \mathbb{F}$. Then we define the *Kronecker product* of A and B to be the $ac \times bd$ block matrix, with blocks of size $c \times d$, where for $1 \leq r \leq a$, $1 \leq s \leq b$, the (r, s)-th block is $a_{r,s}B$.

The next results are direct from the definitions and will be used frequently without further reference.

Lemma 1.8.4. *Let G be a group, and ρ_1, ρ_2 representations of G over the same field $\mathbb{F}$, with corresponding modules V_1 and V_2 respectively. Then the module $V_1 \otimes V_2$ has action group given by $\{(g\rho_1) \otimes (g\rho_2) : g \in G\}$.*

Lemma 1.8.5. *Let A and B be square matrices over the same field $\mathbb{F}$. Then if A is an $n \times n$ matrix and B is an $m \times m$ matrix then $\det(A \otimes B) = \det(A)^m \det(B)^n$.*

We now provide some results about tensor products of representations.

Lemma 1.8.6. *[8, Lemma 1.9.3] Let G be a group, and ρ_1, ρ_2 representations of G over the same field $\mathbb{F}$, with corresponding modules V_1 and V_2 respectively. Then:*

- *For any field automorphism ϕ of $\mathbb{F}$, we have that $(V_1 \otimes V_2)^\phi = V_1^\phi \otimes V_2^\phi$.*
- *$(V_1 \otimes V_2)^* = V_1^* \otimes V_2^*$, where V^* denotes the dual module of V.*
- *$V_1 \otimes V_2 \cong V_2 \otimes V_1$.*

Proposition 1.8.7. *[8, Proposition 1.9.4] Let G be a group, and ρ_1, ρ_2 representations of G over the same field $\mathbb{F}$, with corresponding modules V_1 and V_2 respectively. Suppose that $G\rho_1$ and $G\rho_2$ preserve nondegenerate bilinear forms B_1 and B_2 respectively. Then:*

(i) $G(\rho_1 \otimes \rho_2)$ preserves the bilinear form $B_1 \otimes B_2$.

(ii) If B_1 and B_2 are both symmetric or both anti-symmetric, then $B_1 \otimes B_2$ is symmetric. If one of B_1 or B_2 is symmetric and the other is anti-symmetric then $B_1 \otimes B_2$ is anti-symmetric.

(iii) If $\mathrm{char}(\mathbb{F}) = 2$ and B_1 and B_2 are alternating, then $B_1 \otimes B_2$ is also alternating, and $G(\rho_1 \otimes \rho_2)$ also preserves a quadratic form Q. If $\mathbb{F}$ is finite then Q is of plus type.

1.8.2 Symmetric powers

We provide a very brief definition of the symmetric power of a module; we refer the reader to [8, Section 5.2] for more details.

Definition 1.8.8. Let G be a group, and let V be an n-dimensional module with basis $e_1, \ldots, e_n$. Then the *symmetric power* $S^k(V)$ of V is obtained by quotienting the tensor power $V^{\otimes k}$ by the module $K := \langle (v_1 \otimes \cdots \otimes v_k) - (v_{1^{\sigma^{-1}}} \otimes \cdots \otimes v_{k^{\sigma^{-1}}}) : v_i \in \{e_1, \ldots, e_n\}, \sigma \in \mathrm{Sym}(k) \rangle$.

Remark 1.8.9. It follows from the definition that $S^k(V)$ is generated by elements of the form $e_{i_1} \otimes \cdots \otimes e_{i_k} + K$ with $1 \leq i_1 \leq \ldots \leq i_k \leq n$, and hence has dimension $\binom{n+k-1}{k}$.

1.9 Number Theory

1.9.1 Legendre symbols

In this section, we introduce some of the notions from number theory that we will require for the purposes of this thesis. This material is standard; see for instance [45] for a more thorough introduction to the material.

Definition 1.9.1. Let p be an odd prime, and $a \in \mathbb{Z}$. Then a is a *quadratic residue modulo* p (or a *square modulo* p) if there exists an integer x such that $x^2 \equiv a \mod p$.

If $a \not\equiv 0 \mod p$ then the *Legendre symbol* is given by

$$\left(\frac{a}{p}\right) := \begin{cases} 1 & \text{if } a \text{ is a square modulo } p, \\ -1 & \text{if } a \text{ is not a square modulo } p. \end{cases}$$

If $p|a$ then $\left(\frac{a}{p}\right) = 0$.

Now suppose that q is a positive odd number, so that we can write $q = p_1 \ldots p_s$ for p_i odd primes (not necessarily distinct). Then the *Jacobi symbol* is given by $\left(\frac{a}{q}\right) = \prod_{i=1}^{s} \left(\frac{a}{p_i}\right)$.

Note that the Jacobi symbol is an extension of the Legendre symbol; in particular, when q is a prime the Legendre and Jacobi symbols agree.

We collect together a number of useful results involving the Legendre and Jacobi symbols.

Lemma 1.9.2. *[45, Theorems 3.1, 3.4, 3.6 and 3.8] Let p, q be distinct, odd and positive integers, and $a, b \in \mathbb{Z}$.*

(i) $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$.

(ii) $\left(\frac{a}{pq}\right) = \left(\frac{a}{p}\right)\left(\frac{a}{q}\right)$.

(iii) If p and q are coprime, then $\left(\frac{p^2}{q}\right) = \left(\frac{p}{q^2}\right) = 1$.

(iv) If p and q are coprime, then $\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}$.

Using Lemma 1.9.2, it is straightforward to determine, given a specific integer a, the primes p such that a is a square modulo p. In particular, we collect the following standard results in the below lemma, that we will use from now on without further reference.

Lemma 1.9.3. *Let p be an odd prime. Then:*

(i) $\left(\frac{-1}{p}\right) = \begin{cases} 1 & \textit{if } p \equiv 1 \mod 4, \\ -1 & \textit{if } p \equiv 3 \mod 4. \end{cases}$

(ii) $\left(\frac{2}{p}\right) = \begin{cases} 1 & \textit{if } p \equiv \pm 1 \mod 8, \\ -1 & \textit{if } p \equiv \pm 3 \mod 8. \end{cases}$

(iii) $\left(\frac{3}{p}\right) = \begin{cases} 1 & \textit{if } p \equiv \pm 1 \mod 12, \\ -1 & \textit{if } p \equiv \pm 5 \mod 12. \end{cases}$

(iv) $\left(\frac{5}{p}\right) = \begin{cases} 1 & \textit{if } p \equiv \pm 1 \mod 5, \\ -1 & \textit{if } p \equiv \pm 2 \mod 5. \end{cases}$

We will also be interested in an extension of this notion to algebraic irrationalities.

1.9.2 Algebraic irrationalities

Definition 1.9.4. An *algebraic irrationality* is $\alpha \in \mathbb{C} \setminus \mathbb{Q}$ such that there exists a polynomial $f \in \mathbb{Q}[x]$ such that $f(\alpha) = 0$. If f can be chosen to be irreducible and have degree 2, then we say that α is a *quadratic irrationality.*

It is a standard result that for an algebraic irrationality α there exists a unique irreducible monic polynomial $f \in \mathbb{Q}[x]$ such that $f(\alpha) = 0$; such an f is called the *minimal polynomial* of α.

Definition 1.9.5. For algebraic irrationalities $\alpha_1, \ldots, \alpha_n$, we define the *number field* $\mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ to be the smallest subfield of $\mathbb{C}$ containing $\mathbb{Q}$ and $\alpha_1, \ldots, \alpha_n$.

The notion of a number field will be useful when performing various computations involving algebraic irrationalities in characteristic 0. We require little knowledge of number fields beyond the definition; for more information see [45, Section 9.3].

Definition 1.9.6. The *minimal field of realisation* of an algebraic irrationality α in characteristic p is the smallest field $\mathbb{F}$ of characteristic p such that the minimal polynomial f of α has a root in $\mathbb{F}$.

We use the notation of the ATLAS [12] to describe the algebraic irrationalities we will need in the course of this thesis. We will not generally explicitly define these irrationalities (see [12] or [8, Section 4.2] for more details); for our purposes the information contained in the table below will suffice.

The columns in the table below are as follows:

- 'Irrat' gives the name of the algebraic irrationality α in ATLAS notation.
- 'Real' denotes whether $\alpha \in \mathbb{R}$ or $\alpha \in \mathbb{C} \setminus \mathbb{R}$.
- 'Degree' denotes the degree of the minimal polynomial of α.
- 'Min poly' denotes the minimal polynomial of α. We can compute these in MAGMA.
- 'p-modular reduction' denotes the degree of the minimal field extension of $\mathbb{F}_p$ containing α, which typically involves some congruence on the prime p.

Many of the entries in this table are direct from [8, Table 4.2 and Table 4.3] or [48, Table 2.2.1]. The only additional computation we require is for the irrationalities b_{35}, d_{13}, r_5, y_{15} and y_{36}.

- $b_{35} = \frac{-1+\sqrt{-35}}{2}$. The minimal polynomial of b_{35} is x^2+x+9, b_{35} always exists over $\mathbb{F}_{p^2}$, and exists over $\mathbb{F}_p$ if and only if $\sqrt{-35}$ does, which we can determine from the Legendre symbol when p is odd and via a direct calculation when $p = 2$.
- $y_n = z_n + z_n^{-1}$ where z_n denotes a primitive n-th root of unity. [8, Lemma 4.2.1] shows that for $p \nmid n$, $y_n \in \mathbb{F}_{p^e}$ if and only if $p^e \equiv \pm 1 \mod n$, and so our congruences follow from this (or a direct computation for the finite number of cases where $p \mid n$).
- $r_5 = \sqrt{5}$, and congruences follow directly from the Legendre symbol.
- $d_{13} = z+z^3+z^9$ where z is a primitive 13-th root of unity. Thus $\mathbb{Q}(d_{13}) \subset \mathbb{Q}(z)$, and since the existence of z over $\mathbb{F}_q$ (where q is a power of p) depends on the value of $p \mod 13$, hence so does the existence of d_{13} over $\mathbb{F}_q$. Thus we can determine the congruences in Table 1.1 via a direct computation.

Table 1.1: Table of algebraic irrationalities

Irrat	Real	Degree	Min poly	p-modular reduction
z_3	No	2	x^2+x+1	Deg 1: $p \equiv 0,1$ (3)
				Deg 2: $p \equiv 2$ (3)
b_5	Yes	2	x^2+x-1	Deg 1: $p \equiv 0,1,4$ (5)
				Deg 2: $p \equiv 2,3$ (5)
b_7	No	2	x^2+x+2	Deg 1: $p \equiv 0,1,2,4$ (7)
				Deg 2: $p \equiv 3,5,6$ (7)
b_{11}	No	2	x^2+x+3	Deg 1: $p \equiv 0,1,2,3,4,5,9$ (11)
				Deg 2: $p \equiv 2,6,7,8,10$ (11)
b_{31}	No	2	x^2+x+8	Deg 1: $p \equiv 0,1,2,4,5,7,8,9,10,14,16,18,19,20,25,28$ (31)
				Deg 2: $p \equiv 3,6,11,12,13,15,17,21,22,23,24,26,27,29,30$ (31)
b_{35}	No	2	x^2+x+9	Deg 1: $p \equiv 1,3,4,5,7,9,11,12,13,16,17,27,29,33$ (35)
				Deg 2: $p \equiv 2,6,8,18,19,22,23,24,26,31,32,34$ (35)
i	No	2	x^2+1	Deg 1: $p \equiv 1,2$ (4)
				Deg 2: $p \equiv 3$ (4)
i_2	No	2	x^2+2	Deg 1: $p \equiv 1,2,3$ (8)
				Deg 2: $p \equiv 5,7$ (8)
i_5	No	2	x^2+5	Deg 1: $p \equiv 1,2,3,5,7,9$ (20)
				Deg 2: $p \equiv 11,13,17,19$ (20)
r_3	Yes	2	x^2-3	Deg 1: $p \equiv 1,2,3,11$ (12)
				Deg 2: $p \equiv 5,7$ (12)
r_5	Yes	2	x^2-5	Deg 1: $p \equiv 0,1,4,$ (5)
				Deg 2: $p \equiv 2,3$ (5)
r_6	Yes	2	x^2-6	Deg 1: $p \equiv 1,2,3,5,19,23$ (24)
				Deg 2: $p \equiv 7,11,13,17$ (24)
y_7	Yes	3	x^3+x^2-2x-1	Deg 1: $p \equiv 0,1,6$ (7)
				Deg 2: $p \equiv 2,3,4,5$ (7)
y_9	Yes	3	x^3-3x+1	Deg 1: $p \equiv 1,3,8$ (9)
				Deg 3: $p \equiv 2,4,5,7$ (9)
d_{13}	No	4	$x^4+x^3+2x^2-4x+3$	Deg 1: $p \equiv 0,1,3,9$ (13)
				Deg 2: $p \equiv 4,10,12$ (13)
				Deg 4: $p \equiv 2,5,6,7,8,11$ (13)
y_{15}	Yes	4	$x^4-x^3-4x^2+4x+1$	Deg 1: $p \equiv 1,14$ (15)
				Deg 2: $p \equiv 4,11$ (15)
				Deg 4: $p \equiv 2,3,5,7,8,13$ (15)
y_{36}	Yes	6	$x^6-6x^4+9x^2-3$	Deg 1: $p \equiv 1,3,35$ (36)
				Deg 2: $p \equiv 17,19$ (36)
				Deg 3: $p \equiv 2,11,13,23,25$ (36)
				Deg 6: $p \equiv 5,7,29,31$ (36)

1.9.3 The Möbius function

The Möbius function and the notion of Dirichlet convolution will be useful in Section 2.3.

Definition 1.9.7. The *Möbius function* $\mu(n)$ is a function $\mathbb{N} \to \{1, 0, -1\}$ given by:

(1) $\mu(1) = 1$,

(2) $\mu(n) = 0$ if there exists a prime p such that $p^2|n$,

(3) $\mu(p_1 \dots p_k) = (-1)^k$ if all the primes $p_1, \dots, p_k$ are distinct.

Theorem 1.9.8 (Möbius Inversion Formula)**.** *[22, Theorem 2.66] Let f and g be functions $\mathbb{N} \to \mathbb{C}$ such that for all $n \in \mathbb{N}$,*

$$g(n) = \sum_{d|n} f(d).$$

Then for all $n \in \mathbb{N}$,

$$f(n) = \sum_{d|n} \mu(d) g\left(\frac{n}{d}\right)$$

where μ is the Möbius function.

Definition 1.9.9. Given two functions $f, g : \mathbb{N} \to \mathbb{C}$, we define the *Dirichlet convolution* of f and g, denoted $f * g$, by:

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right).$$

Remark 1.9.10. [31, Proposition 1.8.2]

(1) The definition is clearly symmetric, so that $f * g = g * f$.

(2) Dirichlet convolution is associative, so that $f * (g * h) = (f * g) * h$.

(3) The function e_1 defined by $e_1(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{else,} \end{cases}$ acts as an identity, so that $f * e_1 = f$.

Lemma 1.9.11. *[31, Proposition 2.2.7 and Corollary 2.2.17]*

(i) $\phi = \mathrm{id} * \mu$*, where ϕ is the Euler phi function and μ is the Möbius function.*

(ii) $\mathrm{id} * \mathrm{id}\mu = e_1$*, where id is the identity map.*

1.10 Aschbacher's Theorem

Aschbacher, in 1984, provided a classification of all maximal subgroups of classical groups. We will describe the classes in the theorem below in slightly more detail later.

Theorem 1.10.1 (Aschbacher, [1]). *Let Ω be a quasisimple group equal to one of $\mathrm{SL}_n^{\pm}(q)$, $\mathrm{Sp}_n(q)$ or $\Omega_n^{\epsilon}(q)$, and let G be any group such that $\Omega < G < A$ with A as in Remark 1.6.11. Let M be a maximal subgroup of G. Then M lies in one of the following classes:*

($\mathcal{C}_1$) *subspace stabilisers;*

($\mathcal{C}_2$) *imprimitive wreath products;*

($\mathcal{C}_3$) *extension field groups;*

($\mathcal{C}_4$) *simple tensor products;*

($\mathcal{C}_5$) *subfield groups;*

($\mathcal{C}_6$) *extraspecial types;*

($\mathcal{C}_7$) *wreathed tensor products;*

($\mathcal{C}_8$) *classical types; or*

($\mathcal{S}$) *(or $\mathcal{C}_9$) other almost simple groups.*

There are similarities between Aschbacher's theorem for classical groups, and the O'Nan-Scott theorem for S_n and A_n (see for instance [2, Appendix]). Both give a description of properties that the maximal subgroups must satisfy, but do not (in their original statement) describe precisely what the maximal subgroups are. Both also contain a class consisting of representations or permutation representations respectively of almost simple groups, which is often relatively intractable in general. Both theorems have also had extensive work in determining members of the remaining classes as precisely as possible; for the O'Nan Scott theorem this is done in [40], and for Aschbacher's Theorem a classification of all the groups in classes $\mathcal{C}_1, \ldots, \mathcal{C}_8$ is done in [33]. The first eight classes $\mathcal{C}_1, \ldots, \mathcal{C}_8$ are known as the *geometric-type* subgroups, since each class has some geometric structure associated with it.

We will give approximate definitions of each of these classes. The classes $\mathcal{C}_i$ are formally defined in such a way that the elements of $\mathcal{C}_i$ are maximal, with some possible exceptions in smaller dimensions. For the sake of simplicity the definitions provided here will give a rough idea of the structure of groups in each class and will not determine precisely when such groups are maximal; see [8, Chapter 2] or [33, Chapter 4] for more precise definitions.

We also give examples of groups in each of the classes $\mathcal{C}_i$ when $\Omega = \mathrm{SL}_n(q)$, as well as references to algorithms which detect each Aschbacher class $\mathcal{C}_i$. All of the algorithms mentioned in this section have been implemented in MAGMA. These definitions will not be completely precise; in particular, throughout we will assume that Ω is a classical group, excluding maximal subgroups of almost simple extensions of classical groups. For a full and detailed treatment of the classes for almost simple extensions of Ω, see [8, Chapters 2 and 3] or [33].

Throughout the following sections let $\Omega = \mathrm{SL}_n(q)$ for $q = p^e$ where p is prime.

Definition 1.10.2. Let G be as in Theorem 1.10.1, let $i \in \{1, \ldots 8\}$ and let $M < G$ be in Class $\mathcal{C}_i$ be such that there exists no other group $H \in \mathcal{C}_i$ such that $M \lneq H \lneq G$. Then we refer to M as a *$\mathcal{C}_i$-maximal subgroup*.

Note that M may be $\mathcal{C}_i$-maximal but not maximal, as there may exist a group H in some other Aschbacher class such that $M \lneq H \lneq G$.

1.10.1 Class $\mathcal{C}_1$

Definition 1.10.3. A matrix group K acting on the natural module V is *reducible* if it stabilises a proper non-zero subspace of V.

Definition 1.10.4. A group $M \leq \Omega$ is in *Class* $\mathcal{C}_1$ if it is the normaliser of a G-module W or a pair of G-submodules U and W of V, and is maximal with respect to all such groups.

Example 1.10.5. Let $1 < k < n$, and let $K = \mathrm{SL}_k(q) \times \mathrm{SL}_{n-k}(q)$ denote the block-diagonal matrix group generated by

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & I_{n-k} \end{pmatrix} : a \in \mathrm{SL}_k(q) \right\} \cup \left\{ \begin{pmatrix} I_n & 0 \\ 0 & b \end{pmatrix} : b \in \mathrm{SL}_{n-k}(q) \right\}.$$

Then K stabilises two G-modules: one, W, of dimension k and one of dimension $n - k$. This group, however, is not maximal and hence not in $\mathcal{C}_1$, as it is contained in the full class stabiliser of W. The full class stabiliser of W is generated by K and $I_n + E_{n,1}$, and is $\mathcal{C}_1$-maximal.

Membership of this class can be checked computationally using the MEATAXE algorithm, described for example in [28].

1.10.2 Class $\mathcal{C}_2$

Definition 1.10.6. Let K be an n-dimensional matrix group, with corresponding natural module V. Suppose that there exists a direct sum decomposition of V given by $V_1 \oplus V_2 \oplus \cdots \oplus V_t$ (with $t > 1$), where each V_i is $\frac{n}{t}$-dimensional, the V_i are isometric, and K preserves this direct sum decomposition. Then K is said to be *imprimitive.*

Definition 1.10.7. A group $M < \Omega$ is in *Class* $\mathcal{C}_2$ if M is the stabiliser in Ω of an imprimitive decomposition, and is maximal with respect to all such groups.

For a family of examples of $\mathcal{C}_2$-candidates, we use the following construction. (See for instance [46, p.172] for more details).

Definition 1.10.8. Let D and Q be groups with $Q < S_t$, let $\{D_i : i \in \{1, \ldots, t\}\}$ be a family of isomorphic copies of D, and define $K := \prod_{i=1}^{t} D_i$. Then the *wreath product* of D by Q, denoted by $D \wr Q$ is the (external) semidirect product of K by Q, where Q acts on K by

$$(d_1, \ldots, d_t)^q = (d_{1^{q^{-1}}}, \ldots, d_{t^{q^{-1}}})$$

for all $q \in Q$.

The normaliser of the intersection of the wreath product $\mathrm{GL}_m(q) \wr S_t$ with $\Omega = \mathrm{SL}_{mt}(q)$ is often a $\mathcal{C}_2$-maximal subgroup of Ω.

Groups in this class can be detected by an algorithm of Holt, Leedham-Green, O'Brien and Rees [26] (unless the group is already either semilinear or a tensor product, which it will detect first).

1.10.3 Class $\mathcal{C}_3$

Definition 1.10.9. The action of G on a module V is said to be $\mathbb{F}$-*semilinear* if for all $g \in G$ there exists $\alpha_g \in \mathrm{Aut}(\mathbb{F})$ such that for all $v, w \in V$, $\lambda \in \mathbb{F}$,

$$(v + \lambda w)^g = v^g + \lambda^{\alpha_g} w^g.$$

Remark 1.10.10. If $\alpha_g = \mathrm{Id}_{\mathbb{F}}$ for every $g \in G$, then this is linearity.

We can view the n-dimensional $\mathbb{F}_{q^s}$-vector space $V = \mathbb{F}_{q^s}^n$ as an n^s-dimensional vector space over $\mathbb{F}_q$, which we denote W. Construct α as a $\mathbb{F}_q$-vector space isomorphism between V and W. This gives rise to an embedding of $\Gamma\mathrm{L}_n(q^s)$ inside $\mathrm{GL}_{n^s}(q)$.

Definition 1.10.11. The group $M < \Omega$ is in *Class* $\mathcal{C}_3$ if the natural module V of M is irreducible and there exists a finite field $\mathbb{F}_{q^s}$ such that we can extend the $\mathbb{F}_q$-vector space structure of V to an $\mathbb{F}_{q^s}$-vector space structure of dimension $\frac{n}{s}$ and the action of M on V is $\mathbb{F}_{q^s}$-semilinear, and if M is maximal with respect to all such groups.

For a more precise definition of this embedding, see [8, Definition 2.2.5].

Example 1.10.12. We follow the explicit construction of a group in this class as described in [25, Section 6].

Let V be an m-dimensional vector space over $\mathbb{F}_{q^s}$ with basis $v_1, \ldots, v_m$. Then we can view $\mathbb{F}_{q^s}$ as an s-dimensional $\mathbb{F}_q$-vector space, with basis $\theta_1, \ldots, \theta_s$.

For any $a \in \mathrm{SL}_m(q^s)$, let a act on V, so that $v^a = \sum_{i=1}^{m} \lambda_i v_i$ for $\lambda_i \in \mathbb{F}_{q^s}$; hence we can write $\lambda_i = \sum_{j=1}^{s} \mu_{i,j}\theta_j$ for $\mu_{i,j} \in \mathbb{F}_q$. Thus a also defines a map on the ms-dimensional space over $\mathbb{F}_q$ spanned by the vectors $\theta_j v_i$, giving us a subgroup K of $\mathrm{SL}_{ms}(q)$ isomorphic to $\mathrm{SL}_m(q^s)$. The normaliser of this group K is a $\mathcal{C}_3$-maximal subgroup of $\mathrm{SL}_{ms}(q)$.

The Smash algorithm for detecting groups in this class is described in Holt, Leedham-Green, O'Brien and Rees [27].

1.10.4 Class $\mathcal{C}_4$

Recall the definition of the tensor product and the Kronecker product in Section 1.8.

Definition 1.10.13. A group $G \leq \mathrm{GL}(V \otimes W, \mathbb{F})$ *preserves the tensor product decomposition* if for all $g \in G$ there exist $g_1 \in \mathrm{GL}(V, \mathbb{F}), g_2 \in \mathrm{GL}(W, \mathbb{F})$ such that for all $v \in V, w \in W$, $(v \otimes w)^g = v^{g_1} \otimes w^{g_2}$.

Definition 1.10.14. A group $M < \Omega$ is in *Class* $\mathcal{C}_4$ if it acts on a space $V \otimes W$ which is absolutely irreducible, M preserves this tensor product decomposition, and M is maximal with respect to all such groups.

Example 1.10.15. The $\mathcal{C}_4$-maximal subgroups of $\mathrm{SL}_n(q)$ are precisely the intersection of $\mathrm{SL}_n(q)$ with groups of the form $\mathrm{GL}_{n_1}(q) \otimes \mathrm{GL}_{n_2}(q) = \{g_1 \otimes g_2 : g_1 \in \mathrm{GL}_{n_1}(q), g_2 \in \mathrm{GL}_{n_2}(q)\}$, where $n = n_1 n_2$ with $2 \leq n_1 < \sqrt{n}$.

Detecting groups of this type can be done via an algorithm of Leedham-Green and O'Brien [37, 38].

1.10.5 Class $\mathcal{C}_5$

Definition 1.10.16. A subgroup $H \leq \mathrm{GL}_n(q^e)$ is said to be *subfield* if H is absolutely irreducible and there exists a proper subfield $\mathbb{F}_q$ of $\mathbb{F}_{q^e}$ and a matrix $c \in \mathrm{GL}_n(q^e)$ such that $H^c \leq \langle \mathrm{Z}(\mathrm{GL}_n(q^e)), \mathrm{GL}_n(q) \rangle$.

Remark 1.10.17. Let $\mathbb{F}_q$ be a subfield of $\mathbb{F}_{q^e}$, and let V_e be a vector space of dimension n over $\mathbb{F}_{q^e}$. Then by considering the $\mathbb{F}_q$-span of basis vectors in V_e, we can obtain an embedding of V_1, an n-dimensional vector space over $\mathbb{F}_q$, inside V_e.

Definition 1.10.18. A group $M < \Omega$, a classical group over $\mathbb{F}_{q^e}$, is in *Class* $\mathcal{C}_5$ if it is of the form $N_\Omega(K)Z(\Omega)$, where K is the action group of V_1 as described above, and M is maximal with respect to all such groups. (In particular, if V_e preserves a form, then elements of K also preserve the form).

Remark 1.10.19. All members of Class $\mathcal{C}_5$ are subfield groups.

Example 1.10.20. The normaliser of the group $\mathrm{SL}_n(q)$ is $\mathcal{C}_5$-maximal in $\mathrm{SL}_n(q^e)$ precisely when e is prime. When e is not prime, there exists a nontrivial factor f of e, and then we have the containment $\mathrm{SL}_n(q) < \mathrm{SL}_n(q^f) < \mathrm{SL}_n(q^e)$.

Remark 1.10.21. Note that, although in the example above the $\mathcal{C}_5$-maximal group was written over $\mathbb{F}_q$, $\mathcal{C}_5$-maximal groups can also be conjugates of such groups and thus the entries of the matrix may be in $\mathbb{F}_{q^e}$. Many of the groups we encounter in this thesis which are expressible over $\mathbb{F}_q$ will require conjugation by some matrix c in order to be written over $\mathbb{F}_q$.

An algorithm of Glasby, Leedham-Green and O'Brien [17] detects whether a group can be written over a smaller field.

1.10.6 Class $\mathcal{C}_6$

Definition 1.10.22. A *special* group is a p-group G (for p prime) such that either G is elementary abelian, or $Z(G) = [G,G] = \Phi(G) \cong C_p^k$ for some k, i.e. the centre of G, the commutator subgroup of G, and the Frattini subgroup of G (defined as the intersection of all maximal subgroups of G) are equal and elementary abelian.

A p-group G is *extraspecial* if it is special, and in addition $|[G,G]| = p$.

A p-group G is of *symplectic type* if every characteristic abelian subgroup of G is cyclic; in other words, for any abelian subgroup $H \leq G$ which is not cyclic, there exists an automorphism in $\mathrm{Aut}(G)$ which does not fix H.

Example 1.10.23. The dihedral group D_8 and the quarternion group Q_8 are the two extraspecial groups of order 8.

Remark 1.10.24. In general, a result due to P. Hall fully describes the extraspecial groups (see for example [59, p.83] for a proof); they have order p^{1+2n} for p prime and n positive, and for each such order there are precisely two such groups. They are constructed from central products of extraspecial p-groups of order p^3. For example, if $|G| = 2^{1+2n}$ then G is a central product of either n copies of D_8, or $n-1$ copies of D_8 and one copy of Q_8.

Definition 1.10.25. A group $M < \Omega$ is in *Class* $\mathcal{C}_6$ if Ω has dimension $n = r^m$ for r a prime, and M is the normaliser of either an extraspecial group, or (when $r = 2$) a group of symplectic type, with order r^{1+2m} and exponent $r(2, r)$.

This class can be difficult to detect computationally. An algorithm developed by Niemeyer [44] detects, for r prime, $q \equiv 1 \mod r$ and $G < \mathrm{GL}_r(q)$, whether G normalises an extraspecial r group of order r^3 and exponent r.

1.10.7 Class $\mathcal{C}_7$

Definition 1.10.26. A matrix $g \in \Omega$ *preserves a tensor induced decomposition* $V = V_1 \otimes V_2 \otimes \cdots \otimes V_t$ (for V the natural module of G) if for each i there exist matrices g_i which preserve the bilinear form on V_i (if any), and $\sigma \in S_t$ such that for all $v_i \in V_i$

$$(v_1 \otimes \cdots \otimes v_t)^g = v_{1^{\sigma^{-1}}}^{g_{1^{\sigma^{-1}}}} \otimes \cdots \otimes v_{t^{\sigma^{-1}}}^{g_{t^{\sigma^{-1}}}}.$$

A group $G \leq \mathrm{GL}(V, \mathbb{F})$ *preserves a tensor induced decomposition* $V = V_1 \otimes V_2 \otimes \ldots \otimes V_t$ if it is absolutely irreducible and every $g \in G$ preserves the given tensor induced decomposition (equivalently if every generator of G preserves the tensor induced decomposition).

Note the difference between this definition and Definition 1.10.13 is the relaxed requirement that the group merely preserves the structure of the tensor product; in particular, here we allow the tensor factors to be permuted.

Definition 1.10.27. A subgroup G of $\mathrm{GL}_n(\mathbb{F})$ is a *tensor induced group* if it preserves a tensor induced decomposition $\mathbb{F}^n = V_1 \otimes \cdots \otimes V_t$ with $\dim V_i = m$ for all i (and hence $n = m^t$).

Definition 1.10.28. The group $M < \Omega$ is in *Class* $\mathcal{C}_7$ if its natural module V is absolutely irreducible, M is the stabiliser in Ω of a tensor-induced decomposition (with some restrictions on dimension to rule out overlap with Class $\mathcal{C}_2$) and M is maximal with respect to all such groups.

Example 1.10.29. Let $G = \mathrm{SL}_m(q)$ and $S = S_2 = C_2$. Then we can construct the *tensor wreath product* of G and S, which is generated by matrices $g \otimes I_m$ for g a generator of G, plus the permutation matrix sending row i to row $i + m$ (working modulo m^2) for all i. The normaliser of this group in $\mathrm{SL}_{m^2}(q)$ is $\mathcal{C}_7$-maximal.

An algorithm by Leedham-Green and O'Brien [39] detects whether a group is tensor-induced.

1.10.8 Class $\mathcal{C}_8$

This class consists of those subgroups of a classical group which happen to be a member of a different class of classical groups; i.e. those groups on which it is possible to define a second form.

Definition 1.10.30. A group $M < \Omega$ lies in *Class* $\mathcal{C}_8$ if M is the intersection of Ω with the Γ-group (in the notation of Remark 1.6.11) of another classical group, and is maximal with respect to all such groups.

Example 1.10.31. In general, the group $\mathrm{SL}_n(q)$ has $\mathcal{C}_8$-maximal subgroups which are normalisers of $\Omega_n^\epsilon(q)$ (when q is odd), $\mathrm{SU}_n(q)$ (when q is a square) and $\mathrm{Sp}_n(q)$ (when n is even), with some exceptions for certain small values of n or q.

Many standard algorithms exist in Magma and other computer algebra software packages for detecting forms preserved by a given matrix group. These often follow from computations on the underlying modules.

1.10.9 Class $\mathcal{S}$

This class contains all maximal subgroups which do not fall into any of the other eight classes. Although groups in this class do not share any nice geometric structure as those in the other classes do, there are a number of properties that groups in this class must satisfy, given below.

Definition 1.10.32. We define $G^\infty = \bigcap_{n \geq 0} G^{(n)}$ where $G^{(1)} = [G, G]$ is the derived subgroup of G and inductively, $G^{(n)} = [G^{(n-1)}, G^{(n-1)}]$.

Definition 1.10.33. Let Ω be a classical group over $\mathbb{F}_q$, and G be a group such that $\Omega < G < A$ with A as in Remark 1.6.11. Let $M < G$ and let Z be the subgroup of M consisting of scalar matrices. Then M is in *Class* $\mathcal{S}$ if M/Z is almost simple, and all of the following properties hold:

(i) M does not contain Ω^∞.

(ii) M^∞ acts absolutely irreducibly.

(iii) There does not exist a $g \in \mathrm{GL}_n(q)$ such that $(M^\infty)^g$ is defined over a proper subfield of $\mathbb{F}_q$ modulo scalars.

(iv) The following conditions are satisfied regarding forms preserved by M^∞:

(a) M^∞ preserves a nondegenerate unitary form if and only if q is a square and $\Omega = \mathrm{SU}_n(\sqrt{q})$.

(b) M^∞ preserves a nondegenerate quadratic form if and only if $\Omega = \Omega_n^\epsilon(q)$.

(c) M^∞ preserves a nondegenerate symplectic form and no nondegenerate quadratic form if and only if $\Omega = \mathrm{Sp}_n(q)$.

(d) M^∞ preserves no non-zero classical form if and only if $\Omega = \mathrm{SL}_n(q)$.

Remark 1.10.34. These conditions limit the containment of $\mathcal{S}$ in some of the classes $\mathcal{C}_i$;

- Condition (ii) prevents any containment of Class $\mathcal{S}$ in Class $\mathcal{C}_1$ from the definition. It can also be shown (see [8, p.66]) that condition (ii) is not satisfied by any element of $\mathcal{C}_3$, preventing any containment there also.

- Condition (iii) prevents any containment of Class $\mathcal{S}$ in Class $\mathcal{C}_5$.

- Condition (iv) prevents any containment of Class $\mathcal{S}$ in Class $\mathcal{C}_8$, as this means that no element of $\mathcal{S}$ can preserve more than one type of form.

It is, however, possible for an element of $\mathcal{S}$ to be contained in a subgroup in Class $\mathcal{C}_i$ for $i = 2, 4, 6, 7$.

Class $\mathcal{S}$ splits into two classes, depending on the shape of the nonabelian simple composition factor.

Definition 1.10.35. Let M be a subgroup in class $\mathcal{S}$ of a classical group Ω in characteristic p. Then:

- M is in *Class* $\mathcal{S}_2$ if M^∞ is isomorphic to a group of Lie type in characteristic p. This case is also known as the *defining characteristic case.*

- M is in *Class* $\mathcal{S}_1$ if it is not in Class $\mathcal{S}_2$. This case is also known as the *cross characteristic case.*

We will consider groups in Class $\mathcal{S}_1$ and groups in Class $\mathcal{S}_2$ separately.

1.11 Maximal subgroups of almost simple groups and novelties

We can often determine the maximal subgroups of almost simple extensions of a simple group S from the maximal subgroups of S, although we require some additional work involving class stabilisers. Throughout this section, let G be an almost simple group with socle S.

We first note that maximal subgroups of G which contain S can be determined from the maximal subgroups of the soluble group G/S, and so we will not include these in our classification of maximal subgroups of G. Thus if M is a maximal subgroup of G, we may assume that $M \cap S \lneq S$. In particular, $M \cap S$ is either a maximal subgroup of S, or is contained in a maximal subgroup of S. It also follows from [58, Lemma 2.1] that $M \cap S \neq 1$.

Supppose first that $M \cap S$ is a maximal subgroup of S; in particular it follows that $M \cap S$ is self-normalising in S. Then we can determine the structure of M from the structures of G and $M \cap S$. Note that G is the extension of S by some subgroup H of the outer automorphism group of S. If $h \in G \setminus S$ normalises $M \cap S$ then there exists an outer automorphism α of $M \cap S$ which is induced by h. Then it follows that $(M \cap S).\langle\alpha\rangle < G$, and indeed $(M \cap S).\langle\alpha\rangle$ is a maximal subgroup of $S.\langle h\rangle$. Conversely if h does not normalise $M \cap S$ then it follows similarly that $M \cap S$ does not normalise to a maximal subgroup of $S.\langle h\rangle$.

Hence, if $M < G$ restricts to a maximal subgroup of S, it is relatively straightforward to determine the structure of $M < G$ from the structure of $M \cap S < S$ from the class stabilisers. We now describe the situation where this does not happen.

Definition 1.11.1. If M is a maximal subgroup of G, but $M \cap S$ is not a maximal subgroup of S, then we say that M is a *novelty* maximal subgroup of G.

There are two possible sources of novelty subgroups of G. Let M denote a novelty subgroup of G. Then in particular, $|N_G(M \cap S)| > |M \cap S|$, so there exists $h \in G \setminus S$ which normalises $S \cap M$. For convenience we will now assume that $G = \langle S, h\rangle$.

Firstly, suppose that h normalises $M \cap S$ but not K. Then there exists an automorphism α of $M \cap S$ such that h induces α on $M \cap S$, and we have that $(M \cap S).\langle\alpha\rangle < \langle S, g_h\rangle$. From the above discussion K is a maximal subgroup of $\langle S, g_h\rangle$, but $(M \cap S).\langle\alpha\rangle \not< K$ since $M \cap S$ is self-normalising in S. This motivates one class of novelties:

Definition 1.11.2. Let M be a maximal subgroup of G, an almost simple group with socle S, and suppose that there exists a subgroup $K < S$ such that $(M \cap S) \lneq K \lneq S$. Then M is a *type-1 novelty* maximal subgroup of G with respect to K if there exists $h \in \mathrm{Out}(S)$ and $g_h \in G \setminus S$ such that g_h induces h, normalises $M \cap S$ and does not normalise K.

Next, suppose that h normalises $M \cap S$ and h also normalises K; thus there exist automorphisms α of $M \cap S$ and β of K such that h induces α on $M \cap S$ and β on K. If there exists an element $g_h \in G \setminus S$ such that g_h induces h on S, α on $M \cap S$ and β on K, then we have the containment $M = (M \cap S).\alpha \lneq K.\beta \lneq \langle S, g_h \rangle$ so that M is not a maximal subgroup of G; hence for M to be maximal we must have that no such g_h exists. This motivates the remaining class of novelties:

Definition 1.11.3. Let M be a maximal subgroup of G, an almost simple group with socle S, and suppose that there exists a subgroup $K < S$ such that $(M \cap S) \lneq K \lneq S$. Then M is a *type-2 novelty* maximal subgroup of G with respect to K if there exists $h \in \mathrm{Out}(S)$ and $g_h \in G \setminus S$ such that h normalises $M \cap S$ and K, inducing automorphisms $\alpha \in \mathrm{Aut}(M \cap S)$ and $\beta \in \mathrm{Aut}(K)$ respectively, and g_h normalises $M \cap S$ and induces α, but g_h does not normalise K.

Chapter 2

Determinants and spinor norms of permutation matrices

2.1 Introduction

Suppose we have a classical group G of dimension n over the field $\mathbb{F} = \mathbb{F}_{q^d}$, with the form preserved by G (if such a form exists) being over $\mathbb{F}_p$. The map $\sigma : \mathbb{F} \to \mathbb{F}, a \mapsto a^q$ is an automorphism of $\mathbb{F}$ of order d which extends to an automorphism of G, which we denote σ_G. Let V be the natural module of G, and denote by V^σ the module obtained by applying the automorphism σ_G of G to the associated representation.

Let $\hat{V} = V \otimes V^\sigma \otimes \ldots \otimes V^{\sigma^{d-1}}$, which is a module of dimension n^d. By the construction of $\hat{V}$ we have that $\hat{V}^\sigma \cong \hat{V}$. Thus it follows from Corollary 1.7.20 that we can realise the image of the corresponding representation after conjugation as a subgroup of $\mathrm{GL}_{n^d}(q)$.

Definition 2.1.1. (i) The G-module $\hat{V} = V \otimes V^\sigma \otimes \cdots \otimes V^{\sigma^{d-1}}$ is called the *tensor field representation*. The corresponding action group of $\hat{V}$, denoted $\hat{G} = G \otimes G^\sigma \otimes \cdots \otimes G^{\sigma^{d-1}} < \mathrm{GL}_{n^d}(q^d)$ is called the *tensor field group*.

(ii) After conjugation by a suitable matrix c such that $H := \hat{G}^c < \mathrm{GL}_{n^d}(q)$, we refer to H as a *rewritten tensor field group*, and the corresponding representation of H as a *rewritten tensor field representation*.

(iii) The automorphism σ_G induces an automorphism $\sigma_{\hat{G}}$ on $\hat{G}$. Since a rewritten tensor field group H is isomorphic to $\hat{G}$, there is a corresponding automorphism of H, which we denote ω_H, which acts on H as $\sigma_{\hat{G}}$ does on $\hat{G}$. The automorphism $\sigma_{\hat{G}}$ is induced by conjugation by a matrix, which we denote

g_σ. The corresponding matrix inducing ω_H on H is denoted h_ω. Note that if $\hat{G}^c = H$ it follows that $h_\omega = g_\sigma^c$.

If H is a rewritten tensor field group isomorphic to $\hat{G}$, and $H < \Omega$ for some classical group Ω, then we are interested in whether h_ω lies inside Ω or some extension of Ω by automorphisms; thus we are interested in how h_ω scales the induced form on Ω, the determinant of h_ω and the spinor norm of h_ω with respect to the above form in the case where the form is orthogonal. In this chapter we will answer the question for the related quantity g_σ, in the case where $\hat{G} < \hat{\Omega}$ where $\hat{\Omega}$ is a classical group over $\mathbb{F}_{q^d}$ preserving the same form as Ω. In this case, the matrices g_σ and h_ω will have the same determinant, and we conjecture a connection between their spinor norms with respect to certain forms.

In Section 2.2 we establish conditions whereupon g_σ preserves the form and has determinant 1. Sections 2.3 and 2.4 find respectively the spinor norm and quasideterminant of g_σ in the case where g_σ preserves the orthogonal form $\text{antidiag}(1, \ldots, 1)$. This is the most complicated case, and we will also cover the simpler case where the rewritten tensor field group preserves a symplectic form. Finally Section 2.5 summarises the results.

2.2 Determinants of permutation matrices

In this section, we find the determinant of elements of $\text{Sym}(d)$ acting on $(\mathbb{Z}/n\mathbb{Z})^d$ in the following way. This result can also be found in [33, Equation 4.7.8].

Lemma 2.2.1. *Let $V = (\mathbb{Z}/n\mathbb{Z})^d$ be the set of d-tuples of elements of $\{0, 1, \ldots, n-1\}$, where $n \geq 2$ and $d \geq 2$. Let $\text{Sym}(d)$ act on this set by permuting the elements, so that for $v = (v_1, \ldots, v_d) \in V$, $\sigma \in \text{Sym}(d)$, $v^\sigma = (v_{1^{\sigma^{-1}}}, v_{2^{\sigma^{-1}}}, \ldots, v_{d^{\sigma^{-1}}})$, where i^σ denotes the image of i under the permutation σ; this action gives us an embedding $\text{Sym}(d) \subset \text{Sym}(V) \cong \text{Sym}(n^d)$. We say that the embedding is* even *if $\text{Sym}(d) \subset \text{Alt}(V)$ and* odd *if $\text{Sym}(d) \setminus \text{Alt}(d) \subset \text{Sym}(V) \setminus \text{Alt}(V)$. Then the embedding is even if and only if either $n \equiv 0, 1 \mod 4$, or $n \equiv 2 \mod 4$ and $d > 2$.*

Proof. Let ρ denote the embedding map $\text{Sym}(d) \to \text{Sym}(V)$. The group $\text{Sym}(d)$ is generated by 2-cycles, so it suffices to determine the parity of $\rho((i, j))$ in $\text{Sym}(V)$ for $i, j \in \{1, \ldots, d\}$, $i \neq j$. Take such a 2-cycle; then elements of V will either be fixed or permuted in pairs by this element, and those elements which are fixed are given by $\text{Fix}(\rho((i, j))) = \{v \in V : v_i = v_j\}$ which is a $(\mathbb{Z}/n\mathbb{Z})$-submodule of the $\mathbb{Z}/n\mathbb{Z}$-module V of dimension $d-1$ and size n^{d-1}; hence we have the set of elements which are not fixed by $\rho((i, j))$ has size $\frac{n-1}{n}|V| = n^{d-1}(n-1)$. The orbit of each of these elements has size 2, so that $\rho((i,j))$ consists of $\frac{n^{d-1}(n-1)}{2}$ 2-cycles, and this quantity being odd (respectively even) ensures that the corresponding embedding is odd (respectively even). If n is odd, this parity depends precisely on $n \mod 4$, and the result follows. If $n \equiv 0 \mod 4$ then the term $\frac{n^{d-1}}{2}$ will be even and thus we will have an even embedding; if $n \equiv 2 \mod 4$, then we will have an odd embedding for $d = 2$, and an even embedding for larger d. □

Corollary 2.2.2. *Let $G = \mathrm{SL}_n(q^d)$, where $n \geq 2$ and $q = p^e$, with natural module V and field automorphism $\sigma = \phi^e$. Let $\hat{V} := V \otimes V^\sigma \otimes \ldots \otimes V^{\sigma^{d-1}}$ denote the tensor field representation, with a corresponding tensor field group $\hat{G} < \Omega$, where Ω is the classical group of matrices preserving the induced form of G if G preserves a classical form, and $\Omega = \mathrm{SL}_{n^d}(q^d)$ otherwise. Let $g_\sigma \in \mathrm{GL}_{n^d}(q^d)$ induce the automorphism $\sigma_{\hat{G}}$ on $\hat{G}$ by conjugation. Then:*

(i) g_σ lies in the general group of Ω; and

(ii) g_σ has determinant 1 if and only if either $n \equiv 0, 1 \mod 4$, or $n \equiv 2 \mod 4$ and $d > 2$.

Proof. Since $\sigma_{\hat{G}}$ permutes the tensor factors of $\hat{V}$, $\sigma_{\hat{G}}$ stabilises the representation, and hence by Lemma 1.7.12 g_σ can be realised over $\mathrm{GL}_{n^d}(q^d)$. For the determinant computation, note that g_σ is the permutation matrix corresponding to the permutation considered in Lemma 2.2.1, and the conditions on n follow.

When $n > 2$ G preserves no form and hence neither does $\hat{G}$. This is because if G preserves no form, then V is not isomorphic to V^* or $V^{*\sigma}$. If $\hat{V}$ was self-dual, then in particular we would have $V \cong V^{*\sigma^i}$ for some $i \leq d$, contradicting Theorem 4.1.20; in a similar manner using Theorem 4.1.23 $\hat{G}$ cannot preserve a unitary form. Hence we are done when $n > 2$. When $n = 2$, $\mathrm{SL}_2(q^d) \cong \mathrm{Sp}_2(q^d)$ preserves the matrix $f = \mathrm{antidiag}(-1, 1)$. Since this form is left unchanged by the action of $\sigma_{\hat{G}}$ we have that the form preserved by $\hat{G}$ is the d-fold Kronecker product of f, which we denote $\hat{f}$. Since the image of g_σ is in the conformal group by Lemma 1.7.13, it must act on $\hat{f}$ as multiplication by a scalar, and thus it suffices to see how it acts on a single basis element. Take the natural basis e_0, e_1 of $\mathrm{SL}_2(q^d)$ and consider the effect of g_σ on the element $e_0 \otimes e_0 \otimes \ldots \otimes e_0$. Since g_σ permutes the indices, we see that g_σ acts trivially on $\hat{f}$, hence preserving it. □

Corollary 2.2.3. *Let $G = \Omega_n^\epsilon(q^d)$ or $\mathrm{Sp}_n(q^d)$, where $n \geq 2$ and $q = p^e$, with natural module V and field automorphism $\sigma = \phi^e$. Suppose G preserves a form over $\mathbb{F}_q$. Let $\hat{G} < \Omega$ denote the action group of G on $\hat{V} := V \otimes V^\sigma \otimes \ldots \otimes V^{\sigma^{d-1}}$, where Ω is the classical group preserving the symmetric bilinear form obtained from the*

d-fold Kronecker product of the form preserved by G. Let $g_\sigma \in \mathrm{GL}_{n^d}(q^d)$ induce the automorphism $\sigma_{\hat{G}}$ on $\hat{G}$ by conjugation. Then:

(i) g_σ lies in the general group of Ω; and

(ii) g_σ has determinant 1 if and only if either $n \equiv 0, 1 \mod 4$, or $n \equiv 2 \mod 4$ and $d > 2$.

Proof. Similar to that of Corollary 2.2.2. □

Corollary 2.2.3 can also be deduced from [33, Proposition 4.7.4].

The following result will be useful when considering tensor products of the symplectic group later.

Lemma 2.2.4. *Let n and d be even, and let*

$$X = \{x \in (\mathbb{Z}/n\mathbb{Z})^d : |\{i : x_i \in \{1, \ldots, \frac{n}{2}\}\}| \text{ is odd}\}.$$

Let $\mathrm{Sym}(d)$ act on X as in Lemma 2.2.1, and recall the definitions of even and odd embeddings as given in Lemma 2.2.1. Then this embedding is even unless $d = 2$ and $n \equiv 2 \mod 4$.

Proof. Note that $\mathrm{Sym}(d)$ acts on X, and $|X| = \frac{n^d}{2}$. Consider the action of a 2-cycle in $\mathrm{Sym}(d)$ on X. Then a similar argument to that of Lemma 2.2.1 tells us that there are $\frac{(n-1)n^{d-1}}{4}$ 2-cycles when $d > 2$. When $d = 2$ the action of a 2-cycle in $\mathrm{Sym}(d)$ has no fixed points when acting on X, since X has no elements of the form (i, i); thus in this case we have $\frac{n^2}{4}$ 2-cycles.

Since n is even, when $d > 2$ we have an even number of 2-cycles and hence an even embedding. When $d = 2$, the parity of the embedding is odd if $n \equiv 2 \mod 4$ and even if $n \equiv 0 \mod 4$. □

2.3 Spinor norms of permutation matrices in odd characteristic

In some cases, the tensor field group of a classical group will preserve an orthogonal form. Thus, we will need to determine the spinor norm or quasideterminant of g_σ with respect to the induced form. In this section and the next, we will perform these computations in odd and even characteristic respectively.

Throughout this section let q be odd. We will first give some conditions under which the tensor field groups and rewritten tensor field groups preserve an orthogonal form, and then establish the spinor norm of any permutation matrix preserving a specific antidiagonal form, based on properties of the permutation. The rest of the section will perform these computations when the permutation matrix is g_σ.

2.3.1 General theory and preliminary results

Lemma 2.3.1. *Let $n > 2$ be even, $d \geq 2$ and either $G = \Omega_n^+(q^d, \mathrm{antidiag}(1, \dots 1))$ for any d, or $G = \mathrm{Sp}_n(q^d, \mathrm{antidiag}(1, \dots, 1, -1, \dots, -1))$ with d even. Let $\hat{G}$ be the tensor field group, W be a rewritten tensor representation of G over $\mathbb{F}_q$, and H the corresponding rewritten tensor field group. Then $H < \mathrm{GO}^\epsilon_{n^d}(q)$ where $\epsilon = -$ if $n \equiv 2 \mod 4$ and $d = 2$, and $\epsilon = +$ otherwise.*

Proof. Since the Kronecker product of two symmetric or antisymmetric matrices is symmetric, whilst the Kronecker product of a symmetric and an antisymmetric matrix is antisymmetric (both by Proposition 1.8.7), the conditions on n and d ensure that $\hat{G}$ preserves a symmetric bilinear form of plus type.

There exists a matrix $c \in \mathrm{GL}_{n^d}(q^d)$ such that $c^{-1}\hat{G}c = H$. Let $\hat{f}$ denote the form preserved by $\hat{G}$ (i.e. the d-fold tensor product of the form preserved by G). Then H preserves the form $f := c\hat{f}c^T$. Since all forms of the same type are isometric, we can assume without loss of generality that c has been chosen such that $f = \mathrm{diag}(\eta, 1, \dots, 1)$ where $\eta = 1$ if H preserves a plus-type form, and η is a primitive element of $\mathbb{F}_q^*$ if H preserves a minus-type form (since the dimension of f is a power of n which is even, these forms have the correct type).

Let g_σ be the permutation matrix inducing the σ automorphism on $\hat{G}$, as in Corollary 2.2.2. Take any $a \in \hat{G}$, then by definition of g_σ we have $g_\sigma^{-1} a g_\sigma = a^\sigma$. We also have that $c^{-1}ac = b$ for some $b \in H < \mathrm{GL}_{n^d}(q)$; hence $b^\sigma = b$ and so

$$c^{-1}ac = b = (c^{-1}ac)^\sigma = c^{-\sigma} a^\sigma c^\sigma = c^{-\sigma} g_\sigma^{-1} a g_\sigma c^\sigma.$$

Thus rearranging we have that $g_\sigma^{-1} a g_\sigma = (cc^{-\sigma})^{-1} a (cc^{-\sigma})$. Since a was arbitrary we have $cc^{-\sigma}$ and g_σ both induce the $\sigma_{\hat{G}}$ automorphism. Since the natural module of $\hat{G}$ is absolutely irreducible (see Theorem 4.1.20) , we have that $g_\sigma = \lambda cc^{-\sigma}$ for some $\lambda \in \mathbb{F}_{q^d}$; hence we have $c^{-1} g_\sigma c^\sigma = \lambda I_{n^d}$.

Further, g_σ fixes the first and last rows of the form f (which have the same antidiagonal entry since f is symmetric) so that $g_\sigma f g_\sigma^T = f$. Since $\hat{f}$ and f are both realisable over $\mathbb{F}_q$ and $c\hat{f}c^T = f$ we have that $f = f^\sigma = (c\hat{f}c^T)^\sigma = c^\sigma \hat{f} c^{\sigma T}$. Hence we have

$$(c^{-1}g_\sigma c^\sigma)\hat{f}(c^{-1}g_\sigma c^\sigma)^T = c^{-1}g_\sigma c^\sigma \hat{f} c^{\sigma T} g_\sigma^T c^{-T} = c^{-1}g_\sigma f g_\sigma^T c^{-T} = c^{-1}fc^{-T} = \hat{f}.$$

Thus $c^{-1}g_\sigma c^\sigma = \lambda I_{n^d}$ preserves $\hat{f}$, so $\lambda I_{n^d} \in \mathrm{GO}^+_{n^d}(q^d, \hat{f})$. The only scalar matrices in the orthogonal group are $\pm I_{n^d}$ so that $\lambda = \pm 1$, and so since n is even

$$1 = \det(\lambda I_{n^d}) = \det(c^{-1}g_\sigma c^\sigma) = \det(g_\sigma)\det(c)^{q-1}.$$

Also, since g_σ is a permutation matrix we have $\det(g_\sigma) \in \{1, -1\}$ so that $\det(c)^{q-1} = \det(g_\sigma)$. Further, since the elements in $\mathbb{F}^*_{q^d}$ of order $q-1$ are precisely the elements of $\mathbb{F}^*_q$, it follows that $\det(c)^{q-1} = 1$ if and only if $\det(c) \in \mathbb{F}^*_q$.

We have that $c\hat{f}c^T = f$, and since n is even we have that $\det(f) = \eta$ and $\det(\hat{f}) = 1$, so that $\det(c)^2 = \eta$.

If $\eta = 1$ then $\det(c) \in \mathbb{F}_q$. Then $\det(c)^{q-1} = 1$ so $\det(g_\sigma) = 1$. If η is a primitive element of $\mathbb{F}_q$ then in particular it is not a square, so $\det(c) \notin \mathbb{F}_q$. Thus $\det(c)^{q-1} = -1$, so $\det(g_\sigma) = -1$. Hence H preserves a form of plus-type (respectively minus-type) if the determinant of g_σ is 1 (respectively -1). Corollary 2.2.2 tells us that when n is even, $\det(g_\sigma) = -1$ only when $n \equiv 2 \mod 4$ and $d = 2$, giving the result. □

A result which we will make use of later gives information about the matrix c which is used to rewrite a tensor field group over a smaller field.

Lemma 2.3.2. *Let G be a classical group in dimension n over a field $\mathbb{F}_{q^d}$, with $d \geq 2$. Let $\hat{G} = G \otimes G^\sigma \otimes \cdots \otimes G^{\sigma^{d-1}} < \mathrm{GL}_{n^d}(q^d)$ be the corresponding tensor field group, and $c \in \mathrm{GL}_{n^d}(q^d)$ be such that $\hat{G}^c < \mathrm{GL}_{n^d}(q)$ is a rewritten tensor field group. Then (after multiplication by a scalar) we have that $cc^{-\sigma}$ is a permutation matrix inducing the $\sigma_{\hat{G}}$ automorphism on $\hat{G}$. We also have that* $\det c \in \mathbb{F}_q$ *if* $n \equiv 0, 1 \mod 4$ *or* $n \equiv 2 \mod 4$ *and* $d > 2$, *and* $\det c \notin \mathbb{F}_q$ *but* $\det c^2 \in \mathbb{F}_q$ *otherwise.*

Proof. Let g_σ be the permutation matrix which induces the $\sigma_{\hat{G}}$ automorphism on $\hat{G}$. By noting that c writes elements of $\hat{G}$ as matrices over $\mathbb{F}_q$ which are then fixed by the $\sigma_{\hat{G}}$ automorphism, we have that conjugation by $cc^{-\sigma}$ also induces the $\sigma_{\hat{G}}$ automorphism, in a similar manner to the proof of Lemma 2.3.1. Since the module is absolutely irreducible we thus have that $g_\sigma = \mu cc^{-\sigma}$. Since g_σ is defined over $\mathbb{F}_q$ we have $g_\sigma = g_\sigma^{\sigma^i} = \mu^{q^i} c^{\sigma^i} c^{-\sigma^{i+1}}$, and we also have that $|g_\sigma| = |\sigma| = d$. Hence multiplying these equations together gives that

$$I_{n^d} = g_\sigma^d = \mu^{1+q+\ldots+q^{d-1}} I_{n^d} = \mu^{\frac{q^d-1}{q-1}} I_{n^d}$$

so that $\mu \in \mathbb{F}_q$. Hence there exists $\kappa \in \mathbb{F}_{q^d}$ such that $\kappa^{1-q} = \mu$, and rescaling c by κ means that $cc^{-\sigma} = g_\sigma$ (with c still rewriting $\hat{G}$ over a smaller field); i.e. $\det c^{q-1} = \det g_\sigma$. This κ is the scalar referenced in the statement of the proof, and after the rescaling, we have that $cc^{-\sigma}$. The remaining results follow from Lemma 2.2.1; if $\det g_\sigma = 1$ then $\det c$ has multiplicative order $q-1$ and hence lies inside $\mathbb{F}_q$; otherwise $\det c$ has multiplicative order dividing $2(q-1)$ but not $(q-1)$ and so the result follows. □

Computations suggest that the below conjecture is true.

Conjecture 2.3.3. Let G be a classical group over a field of order q^d preserving a symmetric or anti-symmetric form, and let $\hat{G}$ be the tensor field group $G \otimes G^\sigma \otimes \ldots \otimes G^{\sigma^{d-1}}$, where σ is the Frobenius automorphism obtained by raising each entry of $g \in G$ to the q-th power. Suppose $\hat{G}$ preserves a symmetric bilinear form f. Let $c \in \mathrm{GL}_{n^d}(q^d)$ be such that $\hat{G}^c < \Omega^\epsilon_{n^d}(q)$ is a rewritten tensor field group. Let g_σ be the permutation matrix inducing the σ automorphism on $\hat{G}$, and suppose that c has been chosen such that g_σ^c is also over $\mathbb{F}_q$. Then g_σ and g_σ^c have the same spinor norm over $\mathbb{F}_q$ with respect to the symmetric bilinear forms f and $c^{-1} f c^{-T}$ respectively.

If Conjecture 2.3.3 holds, then the computations that we perform on the permutation matrices inducing σ on the tensor field group would be sufficient to understand the behaviour of matrices inducing σ on the rewritten tensor field group.

Lemma 2.3.4. *Let $q = p^e$ for $p \neq 2$ prime, and $S = \mathrm{Sym}(m)$ where $m = 2n+\delta$, $\delta \in \{0,1\}$. Let $\Omega = \mathrm{GO}^\epsilon_m(q)$ preserve the quadratic form $Q = \mathrm{antidiag}(0,\ldots,0,1,\ldots,1)$ (for m even) or $Q = \mathrm{antidiag}(0,\ldots,0,\frac{1}{2},1,\ldots,1)$ (for m odd) with associated polar form $f = Q + Q^T = \mathrm{antidiag}(1,1,\ldots,1)$, and let $s \in S$ be a permutation whose corresponding permutation matrix in $\mathrm{SL}_m(q)$ preserves f. Then the spinor norm of s with respect to f is 1 if $(-2)^{\epsilon_1}(-1)^{\epsilon_2}$ is a square in $\mathbb{F}_q^*$ and -1 otherwise, with the ϵ_i defined as follows. We have $\epsilon_1 = 1$ if s is odd and 0 if s is even, and $\epsilon_2 = 1$ if $\hat{s}$ is odd and 0 if $\hat{s}$ is even, where $\hat{s} \in \mathrm{Sym}(n)$ is obtained in the following way: for each $i \in \{1,\ldots,n\}$, if $i^s := j \notin \{1,\ldots,n\}$, then multiply s from the right by $(j, m+1-j)$. This gives a permutation $s' \in \mathrm{Sym}(1,\ldots,n) \times \mathrm{Sym}(2n+\delta, 2n-1+\delta,\ldots,n+1+\delta)$; then we define $\hat{s}$ to be the projection of s' onto the first component of this direct product.*

Proof. If $\delta = 1$ then every permutation which fixes the form must in particular have $n+1$ as a fixed point; hence this gives us a permutation in $\mathrm{Sym}(2n)$ which also preserves an antidiagonal form, and thus without loss of generality we may assume $\delta = 0$.

We are interested in permutation matrices which preserve f, which is itself a permutation matrix; thus the group of permutation matrices which preserve f are precisely those permutation matrices whose corresponding permutation lie in the centraliser C of the element $(1,2n)(2,2n-1)\dots(n,n+1)$ in $\mathrm{Sym}(2n)$. It is clear that C is generated by $(x,2n{+}1{-}x)$ for $x \in \{1,\dots,n\}$ and $(y,y{+}1)(2n{+}1{-}y,2n{-}y)$ for $y \in \{1,\dots,n-1\}$.

Let V be the natural module of Ω, with basis $\{v_1,\dots,v_{2n}\}$. We can thus view elements of C as matrices which permute this basis of V and preserve the form of Ω. Fix $x \in \{1,\dots,n\}$ and define $a_x := v_x - v_{2n+1-x}$. Then the reflection through a_x, denoted r_{a_x}, fixes v_i if $i \neq x, 2n+1-x$ and interchanges v_x and v_{2n+1-x}, hence inducing the permutation $(x,2n+1-x)$. Thus $(x,2n+1-x)$ is a reflection, and $a_x f a_x^T = -2$. Hence the spinor norm of $(x,2n+1-x)$ with respect to f depends on whether or not -2 is a square in $\mathbb{F}_q$. Let $M_1 = \{(x,2n+1-x) : x \in \{1,\dots,n\}\}$.

Next, define

$$
\begin{aligned}
b_x^- &= v_x - v_{x+1} + v_{2n-x} - v_{2n+1-x}\\
b_x^+ &= v_x - v_{x+1} - v_{2n-x} + v_{2n+1-x}
\end{aligned}
$$

and corresponding reflections $r_{b_x^+}$ and $r_{b_x^-}$. Note that $b_x^+ f (b_x^-)^T = 0$, and so we have

$$
(v)(r_{b_x^+} r_{b_x^-}) = v - \frac{vf(b_x^+)^T}{b_x^+ Q (b_x^+)^T} b_x^+ - \frac{vf(b_x^-)^T}{b_x^- Q (b_x^-)^T} b_x^- = v - \frac{vf(b_x^+)^T}{2} b_x^+ + \frac{vf(b_x^-)^T}{2} b_x^- .
$$

It follows directly from this that $r_{b_x^+} r_{b_x^-}$ fixes v_i if $i \notin \{x, x+1, 2n-x, 2n+1-x\}$ and interchanges v_x with v_{x+1}, and v_{2n-1} with v_{2n+1-x}; for instance,

$$
\begin{aligned}
&v_x(r_{b_x^+} r_{b_x^-})\\
&= v_x - \frac{1}{2}(v_x - v_{x+1} - v_{2n-x} + v_{2n+1-x}) - \frac{1}{2}(v_x - v_{x+1} + v_{2n-x} - v_{2n+1-x})\\
&= v_{x+1}.
\end{aligned}
$$

Thus $r_{b_x^+} r_{b_x^-}$ induces the permutation $(x,x+1)(2n-x,2n+1-x)$, whose spinor norm depends on whether $(b_x^+ f (b_x^+)^T)(b_x^- f (b_x^-)^T)$ is a square. We have that $b_x^+ f (b_x^+)^T = 4$ and $b_x^- f (b_x^-)^T = -4$, hence the spinor norm depends on whether -1 is a square in $\mathbb{F}_q$. Since conjugation by an element of $\mathrm{GO}_{2n}^{\pm}(q)$ does not change the spinor norm, we also have the same dependency for

$$\begin{aligned}&((x, x+1)(2n-x, 2n+1-x))^{(x+1,x+k)(2n-x,2n+1-k-x)}\\&= (x, x+k)(2n+1-k-x, 2n+1-x)\end{aligned}$$

and so the same dependency for $(x, y)(2n+1-x, 2n+1-y)$ for $x, y \in \{1, \dots, n\}$. Let $M_2 = \{(x, y)(2n+1-x, 2n+1-y) : x, y \in \{1, \dots, n\}\}$.

As noted before, C is generated by M_1 and M_2, and hence we can determine the spinor norm of every element of C from the computations done in this proof, and determining the spinor norm of a given permutation matrix now reduces to writing it as a product of elements in M_1 and M_2. Given that s commutes with the permutation inducing the form, we know that if $i^s = j$ then $(2n+1-i)^s = (2n+1-j)$; hence s' as defined in the statement consists of two components, with $\hat{s}$ acting on the left component (as an element of $\mathrm{Sym}(1, \dots, n)$) and the right component (as an element of $\mathrm{Sym}(2n, \dots, n+1)$) in the same way; in particular s' is even. Also s' is a product of k_2 elements in M_2, and we obtain s from s' by multiplying by k_1 elements of M_1, where $k_i \equiv \epsilon_i \mod 2$. The result on the spinor norm follows. □

Example 2.3.5. Consider the permutation $s = (1, 8)(2, 4, 7, 5) \in \mathrm{Sym}(8)$. This commutes with the permutation $(1, 8)(2, 7)(3, 6)(4, 5)$ and so the corresponding permutation matrix fixes the form $\mathrm{antidiag}(1, \dots, 1)$ induced by this permutation. To find the spinor norm with respect to $\mathrm{antidiag}(1, \dots, 1)$ we see that $\{1, 2, 3, 4\}^s = \{8, 4, 3, 7\}$. Multiplying s from the right by $(1, 8)(2, 7)$ gives the permutation $s' = (2, 4)(5, 7)$. Hence $\epsilon_1 = 0$, $\epsilon_2 = 1$ and so the spinor norm of s has a dependency on whether -1 is a square in $\mathbb{F}_q$; i.e. s has spinor norm 1 if $q \equiv 1 \mod 4$ and -1 if $q \equiv 3 \mod 4$.

Corollary 2.3.6. *Suppose we are in the situation of Lemma 2.3.4, except that the polar form preserved is* $\mathrm{antidiag}(k, k, \dots, k)$ *for some* $k \in \mathbb{F}_q$*. Then the spinor norm of* s *with respect to this form for* $p \neq 2$ *depends on whether* $(-2k)^{\epsilon_1}(-1)^{\epsilon_2}$ *is a square in* $\mathbb{F}_q$*, with* ϵ_i *as above.*

Proof. We can construct a_x and $b_x^{\pm}$ in the same way as in Lemma 2.3.4. These reflections induce the same permutations, but now $a_x f a_x^T = -2k$, $b_x^+ f (b_x^+)^T = 4k$ and $b_x^- f (b_x^-)^T = -4k$. Hence the spinor norm of elements of M_2 are unchanged, but the spinor norm of elements of M_1 now depend on $-2k$ instead of -2. The rest of the proof is identical. □

Definition 2.3.7. Let C be the centraliser in $\mathrm{Sym}(m)$ of the permutation mapping

i to $m+1-i$ for all i, and let $c \in C$. Write $m = 2n + \delta$ for $\delta \in \{0, 1\}$. We say that $j \in \{1, \dots, n\}$ is a *crossing* (with respect to c) if $j^{c^{-1}} \notin \{1, \dots, n\}$.

The *dual* of $i \in \{1, \dots, m\}$ is $m + 1 - i$, denoted i^*. The *dual* of the cycle $(a_1, \dots, a_d)$ is $(a_1^*, \dots, a_d^*)$.

An element $j \in \{1, \dots, n\}$ is said to be *self-dual with respect to c* (or just self-dual if c is clear from the context) if $j^c = j^*$. (In particular any self-dual element is a crossing).

A cycle is *self-dual* if it is of the form $(a_1, \dots, a_e, a_1^*, \dots, a_e^*)$, where $e \leq n$.

2.3.2 Spinor norms when d is odd

When d is odd, the required computations are straightforward. Recall the notation introduced at the start of this chapter.

Lemma 2.3.8. *Let $G = \mathrm{GO}_n^\epsilon(q^d)$ with $d > 2$ odd, and G preserving the form* $\mathrm{antidiag}(1, \dots, 1)$ *with natural module V, and $H < \mathrm{GO}_{n^d}^\epsilon(q)$ be a rewritten tensor field group. Let $h_\omega \in \mathrm{CGO}_{n^d}^\epsilon(q)$ induce the ω_H automorphism on H by conjugation. Then h_ω has spinor norm 1 with respect to the form* $\mathrm{antidiag}(1, \dots, 1)$ *over $\mathbb{F}_q$.*

Proof. By Corollary 2.2.2 we know that $h_\omega \in \mathrm{GO}_{n^d}^\epsilon(q)$, which has a subgroup of index 2 consisting of elements with spinor norm 1. Since h_ω is of odd order it must lie in this index 2 subgroup and hence have spinor norm 1. □

2.3.3 Spinor norms when d is even

The aim of this section is to find the spinor norm of the field automorphism of the rewritten tensor field representation. To do this we will consider a closely-related combinatorial problem. Throughout this subsection d will be even unless otherwise stated.

Definition 2.3.9. A *string* of d beads and n colours is a d-tuple where each entry takes one of n possible colourings (usually the set $\{1, \dots, n\}$).

Defne an action of $C_d = \langle \theta \rangle$ on the set of all strings of d beads and n colours by $(a_1, \dots, a_d)^\theta = (a_d, a_1, \dots, a_{d-1})$. A *necklace* of d beads and n colours is an orbit of a string, although we will often abuse notation and also use the term necklace to refer to an orbit representative .

The integer d is referred to as the *length* of the necklace. The *orbit size* of the necklace is the size of the orbit. A necklace of length d and orbit size d is called *aperiodic of orbit size d.*

Example 2.3.10. An example of a string of 6 beads and 3 colours is $(1,2,3,1,2,3)$. Formally, the necklace containing this string is

$$\{(1,2,3,1,2,3),(2,3,1,2,3,1),(3,1,2,3,1,2)\}.$$

With our abuse of notation we will say that all three elements of the orbit are the same necklace. This necklace has length 6 and orbit size 3. An example of an aperiodic necklace of orbit size 6 is $(1,2,3,1,3,2)$.

There is an obvious bijection between the basis elements $v_{a_1} \otimes \ldots \otimes v_{a_d}$ of the rewritten tensor representation and strings of d beads and n colours, and this gives us a bijection between the orbits under the σ automorphism and necklaces of d beads and n colours. Thus, understanding the structure of the permutation that σ induces is equivalent to understanding the total number of necklaces and their orbit sizes. We can also phrase the quantities ϵ_1 and ϵ_2 in Lemma 2.3.4 in terms of combinatorial properties of these necklaces.

Definition 2.3.11. Let $g \in \mathrm{Sym}(n)$ be a permutation, and write $s = a_1 \ldots a_m$ as a disjoint product of cycles. Then by a *factor* of g we mean a permutation of the form $\prod_{i \in I} a_i$ for some $I \subset \{1,\ldots,m\}$.

Lemma 2.3.12. *Let $g \in C < \mathrm{Sym}(n^d)$ where C is the centraliser of the permutation mapping i to n^d+1-i for all i. Let c be a factor consisting of a single l-cycle in g. Let g' be as in Lemma 2.3.4. Then:*

- *If c is not self-dual, then denote its dual by c^*. Then the factor of g' corresponding to cc^* will also consist of two l-cycles.*
- *If c is self-dual, then l is even and the corresponding factor of g' will consist of two $\frac{l}{2}$-cycles.*

Proof. Suppose c is not self-dual. Write $c = (a_1,\ldots,a_l)$, so that we have $cc^* = (a_1,\ldots,a_l)(a_1^*,\ldots,a_l^*)$. (Note that if c is a factor of g then since $g \in C$ we must have that c^* is also a factor of g). The corresponding factor of g', which we denote $(cc^*)'$, is obtained by multiplying cc^* by elements of the form (a_i, a_i^*). Thus, assuming without loss of generality that $a_1 < \frac{n^d}{2}$, we have that the orbit of a_1 under g' is $\{\min\{a_i, a_i^*\} : i = 1,\ldots,l\}$ and the orbit of a_1^* is $\{\max\{a_i, a_i^*\} : i = 1,\ldots,l\}$. Thus $(cc^*)'$ consists of two l-cycles.

If c is self-dual and centralises C, then we can write $c = (a_1,\ldots,a_l,a_1^*,\ldots,a_l^*)$; in particular l must be even. Again, to obtain c' we multiply c by elements of the form (a_i, a_i^*), so that c' maps a_i to $\min\{a_{i+1}, a_{i+1}^*\}$ if $a_i < a_i^*$, otherwise c' maps a_i to $\max\{a_{i+1}, a_{i+1}^*\}$ (reading subscripts modulo l). Thus c' consists of two $\frac{l}{2}$-cycles. $\square$

Remark 2.3.13. Lemma 2.3.12 tells us that, given a permutation g, the only change in the structure of the permutation g' will be that every self-dual even cycle which is a factor of g will split into two cycles of half the length in g'. Thus understanding which cycles in g are self-dual is sufficient to understand the structure of g'.

Theorem 2.3.14 (Burnside's Lemma). *[46, Theorem 3.22] Let G be a finite group acting on a set X. Then the number of orbits of X under G is $\frac{1}{|G|}\sum_{g\in G}|\mathrm{Fix}_X(g)|$.*

Corollary 2.3.15 (Polya Enumeration Theorem). *[46, p. 61] Let X be a finite set and $G < \mathrm{Sym}(X)$. Let Y be a finite set of colours, with $|Y| = n$. Denote the set of all colourings of X with Y by Y^X. Then the number of orbits under G on the set Y^X is $\frac{1}{|G|}\sum_{g\in G} n^{c(g)}$, where $c(g)$ is the number of cycles for $g \in G$ as a permutation of X.*

Definition 2.3.16. A string of d beads choosing from n colours is said to be *self-dual* if the colouring is of the form $(a_1, \dots, a_{d/2}, n+1-a_1, \dots, n+1-a_{d/2})$ for $a_i \in \{1, \dots, n\}$. If n is odd we also consider the necklace $(\frac{n+1}{2})$ of length 1 to be self-dual.

Remark 2.3.17. Hence there is at most one self-dual necklace of odd length, which has length 1 and occurs if and only if n is odd.

Lemma 2.3.18. *The number of necklaces of d beads and n colours of orbit size e (with $e|d$) is the same as the number of aperiodic necklaces of e beads and n colours.*

Proof. For every aperiodic necklace of e beads and n colours, say $a = (a_1, \dots, a_e)$, there is a corresponding necklace of d beads and n colours with orbit size e, namely the concatenation of $\frac{d}{e}$ copies of a. Conversely, suppose we have a necklace of d beads and n colours with orbit size e, say $b = (b_1, \dots, b_d)$. Since the necklace has orbit size e we must have $b_i = b_{i+me}$ for every $i \in \{1, \dots, d\}$ and $m \in \mathbb{Z}$, reading subscripts modulo d; further, e is the smallest possible such integer. Hence we have b as the $\frac{d}{e}$-fold concatenation of $(b_1, \dots, b_e)$, and this necklace of length e must be aperiodic. $\square$

Recall the definition of the Möbius function from Section 1.9.3.

Lemma 2.3.19. *Suppose we have a necklace of d beads which we aim to colour with n colours. Let $\delta \in \{0, 1\}$ denote the parity of n. Then the following formulae hold:*

(i) $\psi(n,d) := \frac{1}{d}\sum_{e|d} \phi(\frac{d}{e}) n^e$ *counts the total number of necklaces, where* ϕ *is the Euler phi function.*

(ii) $\chi(n,d) := \sum_{e|d} \mu(e)\psi(n,\frac{d}{e})$ *counts the number of aperiodic necklaces of orbit size* d*, where* μ *is the Möbius function.*

(iii) $\chi'(n,d) := \frac{1}{d}\left(-\iota(d)\delta + \sum_{e|d, e \text{ odd}} \mu(e) n^{d/2e}\right)$ *where* $\iota(d) = 1$ *if* d *is a power of 2 and 0 otherwise; this counts the number of self-dual aperiodic necklaces of orbit size* d*.*

Proof. Part (i) is a direct application of Polya's enumeration theorem (Corollary 2.3.15) with $G = C_d$, using the fact that the number of elements of C_d of order e (with $e|d$) is $\phi(e)$. For (ii), it follows from Lemma 2.3.18 that

$$\psi(n,d) = \sum_{e|d} \chi(n,e),$$

by summing over all the possible orbit sizes of a necklace of length d. The formula for $\chi(n,d)$ follows from the Möbius inversion formula (Theorem 1.9.8). Note that neither of these equations require d to be even, although the interpretation of $\chi(n,d)$ only makes sense when d is even.

To prove (iii), we introduce some additional notation. Let $\psi'(n,d)$ count the total number of self-dual strings of d beads and n colours. Let $a = (a_1, \ldots, a_d)$ be one such string. We may choose the first $\frac{d}{2}$ of the beads to be any of the n colours, and then the second $\frac{d}{2}$ beads must be the dual colour; in other words, if $a_i = j$ then $a_{i+\frac{d}{2}} = n+1-j$. Thus $\psi'(n,d) = n^{d/2}$. We may also write

$$\psi'(n,d) = \sum_{e|d,\ e \text{ odd}} \kappa(n, \frac{d}{e}),$$

for some sensible choice of function $\kappa(n,\frac{d}{e})$ (which will be defined later), in a manner similar to the relationship between ψ and χ. This follows for reasons similar to those given in Lemma 2.3.18, since there is a one-to-one correspondence between self-dual strings of d beads and n colours of orbit size $\frac{d}{e}$, and self-dual aperiodic strings of $\frac{d}{e}$ beads and n colours; and the sum is over odd factors of d since the above correspondence gives rise to self-dual necklaces if and only if $\frac{d}{e}$ is even (in order for a self-dual necklace of length $\frac{d}{e}$ to exist) and e is odd (so that the corresponding necklace of length d is also self-dual).

The naturally analagous definition of $\kappa(n, \frac{d}{e})$ would be the number of self-dual aperiodic strings of $\frac{d}{e}$ beads and n colours; however, when n is odd, this definition will mean that the self-dual string $(\frac{n+1}{2}, \ldots, \frac{n+1}{2})$ occurring as a d-fold concatenation of the self-dual string of length 1 is not counted in $\psi'(n, d)$. Notice that in the formula for $\psi'(n, d)$ there is always precisely one summand of the form $\kappa(n, 2^r)$, namely when $e = q$ for $d = 2^r q$. Thus the correct definition of $\kappa(n, \frac{d}{e})$ is to count the number of self-dual aperiodic strings of $\frac{d}{e}$ beads and n colours, except where $\frac{d}{e}$ is a power of 2, where it counts the number of self-dual aperiodic strings of n colours and either 1 bead or $\frac{d}{e}$ beads.

Möbius inversion then gives us that

$$\kappa(n, d) = \sum_{e|d, e \text{ odd}} \mu(e)\psi'\left(n, \frac{d}{e}\right) = \sum_{e|d, e \text{ odd}} \mu(e) n^{d/2e}.$$

Now note that $\kappa(n, d) - \iota(d)\delta$ counts the number of self-dual aperiodic strings of d beads and n colours, where ι and δ are as defined in the statement of the lemma, and hence $\chi'(n, d) = \frac{1}{d}(\kappa(n, d) - \iota(d)\delta) = \frac{1}{d}\left(-\iota(d)\delta + \sum_{e|d, e \text{ odd}} \mu(e) n^{d/2e}\right)$. □

We now consider the permutation s inducing the field automorphism σ.

Lemma 2.3.20. *Let $s \in \mathrm{Sym}(n^d)$ be the permutation defined by mapping $n^{d-1}i_1 + n^{d-2}i_2 + \ldots + ni_{d-1} + i_d + 1$ to $n^{d-1}i_2 + n^{d-2}i_3 + \ldots + ni_d + i_1 + 1$ (i.e. permuting digits in the n-ary expansion cyclically). The number of self-dual cycles c of length e which are factors of s is $\chi'(n, e)$.*

Proof. For c to be an e-cycle in s, we must have $e|d$ and if an orbit representative for c is given by $n^{d-1}i_1 + n^{d-2}i_2 + \ldots + ni_{d-1} + i_d + 1$ then $i_{k+e} = i_k$ for all k. If c is a self-dual cycle, we must in addition have that e is even and $i_k + i_{k+\frac{e}{2}} = n + 1$. Thus, the corresponding string $(i_1, \ldots, i_e, i_1, \ldots, i_e, \ldots, i_1, \ldots, i_e)$ is a concatenation of self-dual aperiodic necklaces of orbit size e.

Conversely, given a self-dual aperiodic necklace of orbit size e, say $(i_1, \ldots, i_e)$, concatenating $\frac{d}{e}$ of these together and using these as the coefficients of the n-ary expansion of some integer in the set $\{1, \ldots, n^d\}$, we obtain an orbit representative for an e-cycle contained in c, and this e-cycle is independent of the choice of string $(i_1, \ldots, i_e)$.

Thus we have the same number of factors of s which are e-cycles and self-dual aperiodic necklaces of orbit size e; by definition $\chi'(n, e)$ counts the latter and so the result follows. □

Corollary 2.3.21. *Let $s \in \mathrm{Sym}(n^d)$ be as in Lemma 2.3.20 and let s' be as in Lemma 2.3.4. If $e|d$ then the number of e-cycles in s' is given by $X(n,e,d) = \chi(n,e) + \delta^+ 2\chi'(n,2e) - \delta^- \chi'(n,e)$ where $\delta^+ = 1$ if $2e|d$ and 0 otherwise, and $\delta^- = 1$ if $2|e$ and 0 otherwise*

Proof. Notice that by Remark 2.3.13 the only change in the structure of s' compared to s occurs when e is even, when every self-dual e-cycle splits into two $\frac{e}{2}$ cycles. Since $\chi'(n,e)$ counts the number of self-dual e-cycles of s by Lemma 2.3.20, the claim follows. □

We retain the notation of Lemma 2.3.19 and Corollary 2.3.21 for the rest of this section.

The following formula is well-known, but no convenient reference could be found for it.

Lemma 2.3.22 (Moreau's necklace-counting formula)**.** $\chi(n,d) = \frac{1}{d}\sum\limits_{e|d} \mu(e) n^{d/e}$.

Proof. We can rewrite:

$$\begin{aligned}
\chi(n,d) &= \sum_{e|d} \mu\left(\frac{d}{e}\right) \psi\left(n,e\right) && \text{interchanging } e \text{ and } \frac{d}{e} \\
&= \sum_{e|d} \frac{1}{e} \mu\left(\frac{d}{e}\right) \sum_{f|e} \phi\left(\frac{e}{f}\right) n^f \\
&= \frac{1}{d} \sum_{e|d} \frac{d}{e} \mu\left(\frac{d}{e}\right) \sum_{f|e} \phi\left(\frac{e}{f}\right) n^f.
\end{aligned}$$

The coefficient of n^f is thus given by

$$\frac{1}{d} \sum_{f|e|d} \frac{d}{e} \mu\left(\frac{d}{e}\right) \phi\left(\frac{e}{f}\right) = \frac{1}{d} \sum_{h|\frac{d}{f}} \frac{d}{hf} \mu\left(\frac{d}{hf}\right) \phi(h)$$

with $h = \frac{e}{f}$, and so adjusting the limits in the sums we have

$$
\begin{aligned}
\chi(n,d) &= \frac{1}{d}\sum_{f|d}\left(\sum_{h|\frac{d}{f}}\frac{d}{hf}\mu\left(\frac{d}{hf}\right)\phi(h)\right)n^{f} \\
&= \frac{1}{d}\sum_{f|d}\left(\sum_{h|f}\frac{f}{h}\mu\left(\frac{f}{h}\right)\phi(h)\right)n^{d/f} && \text{interchanging } f \text{ and } \frac{d}{f} \\
&= \frac{1}{d}\sum_{f|d}\left(\sum_{h|f}h\mu(h)\phi\left(\frac{f}{h}\right)\right)n^{d/f} && \text{interchanging } h \text{ and } \frac{f}{h}.
\end{aligned}
$$

Hence the coefficient of $n^{d/f}$ is given by the Dirichlet convolution $(\mathrm{id}\mu * \phi)(f)$. By Lemma 1.9.11 $\phi = \mathrm{id} * \mu$, so that the coefficient of $n^{d/f}$ is

$$
(\mathrm{id}\mu * \mathrm{id} * \mu)(f) = (e_1 * \mu)(f) = \mu(f).
$$

Hence we have

$$
\chi(n,d) = \frac{1}{d}\sum_{f|d}\mu(f)n^{d/f}
$$

as required. □

Lemma 2.3.23. $\chi(n,d) = 2\chi'(n,2d) - \chi'(n,d)$.

Proof.

$$
\begin{aligned}
&2\chi'(n,2d) - \chi'(n,d) \\
&= \frac{2}{2d}\left(-\iota(2d)\delta + \sum_{e|2d, e \text{ odd}}\mu(e)n^{2d/2e}\right) - \frac{1}{d}\left(-\iota(d)\delta + \sum_{e|d, e \text{ odd}}\mu(e)n^{d/2e}\right) \\
&= \frac{1}{d}\left(\sum_{e|2d, e \text{ odd}}\mu(e)n^{d/e} - \sum_{e|d, e \text{ odd}}\mu(e)n^{d/2e}\right) \text{(since } \iota(2d) = \iota(d)) \\
&= \frac{1}{d}\left(\sum_{e|d, e \text{ odd}}\mu(e)n^{d/e} - \sum_{e|d, e \text{ odd}}\mu(e)n^{d/2e}\right)
\end{aligned}
$$

(since odd factors of d and $2d$ are the same)

$$= \frac{1}{d}\left(\sum_{e|d, e \text{ odd}} \mu(e)n^{d/e} + \sum_{e|d, e \text{ odd}} \mu(2)\mu(e)n^{d/2e}\right)$$

$$= \frac{1}{d}\left(\sum_{e|d, e \text{ odd}} \mu(e)n^{d/e} + \sum_{e|d, e \text{ odd}} \mu(2e)n^{d/2e}\right) \text{ (since } \mu \text{ is multiplicative)}$$

$$= \frac{1}{d}\left(\sum_{e|d, e \text{ odd}} \mu(e)n^{d/e} + \sum_{e|d, e \text{ even}} \mu(e)n^{d/e}\right) \text{ (since } \mu(2^r e) = 0 \text{ for } r > 1)$$

$$= \frac{1}{d}\left(\sum_{e|d} \mu(e)n^{d/e}\right) = \chi(n, d).$$

□

We now prove results on the congruences of $\chi(n, d)$ and $\chi'(n, d)$ modulo 4, which are sufficient to determine ϵ_1 and ϵ_2 (in the notation of Lemma 2.3.4) for the permutation s.

Lemma 2.3.24. *Let n be a positive integer and $t \geq 0$ be such that $n \equiv \pm 1 \mod 2^t$ and $n \not\equiv \pm 1 \mod 2^{t+1}$. Then for every positive integer r, $n^{2^r} \equiv 1 \mod 2^{t+r}$ and $n^{2^r} \not\equiv 1 \mod 2^{t+r+1}$.*

Proof. Induct on r. By induction (or assumption in the base case) we can write $n^{2^r} \equiv \theta_0 + \theta_1 2^{t+r+1} + \delta\theta_2 2^{t+r+2} \mod 2^{t+r+3}$, for $\theta_i \in \{-1, 1\}$ (with $\theta_0 = 1$ except when $r = 1$), $\delta \in \{0, 1\}$. We then obtain that $n^{2^{r+1}} \equiv 1 + \theta_0\theta_1 2^{t+r+2} \mod 2^{t+r+3}$, so that $n \equiv 1 \mod 2^{t+r+2}$ and $n \not\equiv 1 \mod 2^{t+r+3}$. □

In particular, we will use the following congruences frequently:

Corollary 2.3.25. *Let n be odd. Then for all $r \geq 1$:*

(i) $n^{2^r} \equiv 1 \mod 2^{r+2}$.

(ii) If $n \equiv \pm 1 \mod 8$ *then* $n^{2^r} \equiv 1 \mod 2^{r+3}$.

(iii) If $n \equiv \pm 3 \mod 8$ *then* $n^{2^r} \equiv 2^{r+2} + 1 \mod 2^{r+3}$.

Corollary 2.3.26. *Suppose n is odd, $n > 1$. Then $\chi(n, d) \equiv 0 \mod 4$ if d is not a power of 2 or if $n \equiv \pm 1 \mod 8$, and $\chi(n, d) \equiv 2 \mod 4$ if $d > 2$ is a power of 2* *and* $n \equiv \pm 3 \mod 8$. *We also have* $\chi(n,2) \equiv \begin{cases} 0 \mod 4 & \text{if } n \equiv 1 \mod 8, \\ 3 \mod 4 & \text{if } n \equiv 3 \mod 8, \\ 2 \mod 4 & \text{if } n \equiv 5 \mod 8, \\ 1 \mod 4 & \text{if } n \equiv 7 \mod 8. \end{cases}$

Proof. From Lemma 2.3.22 we have $\chi(n,d) = \frac{1}{d}\sum_{e|d}\mu(e)n^{d/e}$. Write $d = 2^r q$ for q odd, and suppose first that $q \neq 1$. We are interested in the value of $\chi(n,d) \mod 4$, but it is more convenient to find the value of $d\chi(n,d) \mod 4d$. Additionally, since we know that $\chi(n,d)$ is an integer and $q|d$, we must have $d\chi(n,d) \equiv 0 \mod q$ and so it is sufficient to work modulo 2^{r+2}, in which case we have the following, using Lemma 2.3.22:

$$\begin{aligned}
d\chi(n,d) &= \sum_{e|d}\mu(e)n^{d/e} \\
&\equiv \sum_{e|d,e \text{ even}}\mu(e)n^{d/e} + \sum_{e|d,e \text{ odd}}\mu(e) \quad \text{by Corollary 2.3.25(i)} \\
&= \sum_{e|d,e \text{ even}}\mu(e)n^{d/e} \quad \text{as} \sum_{e|d,e \text{ odd}}\mu(e) = \sum_{e|q}\mu(e) = e_1(q) = 0 \text{ for } q \neq 1 \\
&= -\sum_{e|q}\mu\left(e\right)n^{2^{r-1}q/e} \quad \text{since } \ \mu(e) = 0 \text{ if } 4|e \\
&\equiv -n^{2^{r-1}}\sum_{e|q}\mu(e) \quad \text{reducing powers of } n \text{ modulo } 2^r \\
&= 0 \quad \text{since } q \neq 1.
\end{aligned}$$

If $q = 1$ then $d\chi(n,d) = n^{2^r} - n^{2^{r-1}}$ and working modulo 2^{r+2} the result follows directly from Corollary 2.3.25 when $r > 1$. When $d = 2$ we have $\chi(n,2) = \frac{n(n-1)}{2}$ and the result follows by standard calculations. □

Lemma 2.3.27. *Suppose* $n \equiv 0 \mod 4$, *and* $d = 2^r q$ *where* q *is odd. Then* $\chi(n,d) \equiv 2 \mod 4$ *if* $n \equiv 4 \mod 8$ *and* d *is squarefree, and otherwise* $\chi(n,d) \equiv 0 \mod 4$.

Proof. Suppose first that $r = 1$, and so as in Corollary 2.3.26 we are interested in $\sum_{e|d}\mu(\frac{d}{e})n^e \mod 8$. If $n \equiv 0 \mod 8$ then so is the whole sum, so suppose $n \equiv 4 \mod 8$. Then the only term in the sum which will not be divisible by 8 is $\mu(d)n$, which will be $4 \mod 8$ precisely when d is squarefree.

Now suppose $r > 1$, so we are interested in $\sum_{e|d}\mu(\frac{d}{e})n^e \mod 2^{r+2}$. Since $n \equiv 0 \mod 4$ any summand n^e not congruent to 0 modulo 2^{r+2} must satisfy $4^e < 2^{r+2}$, i.e.

$2e < r+2$. We also require $\mu(\frac{d}{e}) \neq 0$, so that $2^{r-1}|e$. In particular we must have $2^r < r+2$, which is not satisfied for any integer $r > 1$. Hence all terms in the sum are $0 \mod 2^{r+2}$. $\square$

Lemma 2.3.28. *Suppose $n \equiv 2 \mod 4$, and $d = 2^r q$ where q is odd. Then:*

$$\chi(n,d) \equiv \begin{cases} \frac{n}{2} q\mu(q) \mod 4 & \text{if } r = 1, \\ -q\mu(q) \mod 4 & \text{if } r = 2, \\ 2\mu(q) \mod 4 & \text{if } r = 3, \\ 0 \mod 4 & \text{if } r \geq 4. \end{cases}$$

Proof. Similarly to Lemma 2.3.27, any summand n^e in $d\chi(n,d) = \sum_{e|d} \mu(d/e) n^e$ which is not congruent to $0 \mod 4$ must satisfy $e < r+2$ (since $n \equiv 2 \mod 4$) and $2^{r-1}|e$; in particular we must have $2^{r-1} < r+2$. This does not hold for $r \geq 4$, so the last condition in the statement holds.

For the other cases, we are interested in the value modulo 2^{r+2} of $d\chi(n,d) = \sum_{e|d} \mu(d/e) n^e$. We consider the possible values of r separately.

For $r = 3$, the summands $\mu(d/e)n^e$ which are not equivalent to 0 modulo 32 satisfy $e < 5$ and $4|e$. Thus $d\chi(n,d) \equiv \mu(2q)n^4 \equiv 16\mu(2q) \mod 32$; hence $\chi(n,d) \equiv 0 \mod 4$ if q is not squarefree and $\chi(n,d) \equiv 2 \mod 4$ if q is squarefree. Note here that since $\chi(n,d)$ is even we have $\mu(2q)n^4 = \mu(q)n^4$.

For $r = 2$, we require $e < 4$ and $2|e$, so we have $d\chi(n,d) \equiv 4\mu(2q) = -4\mu(q)$ modulo 16; thus if q is not squarefree we have $\chi(n,d) \equiv 0 \mod 4$. Otherwise we have $q\chi(n,d) \equiv -\mu(q) \mod 4$ and so $\chi(n,d) \equiv -q^{-1}\mu(q) \equiv -q\mu(q) \mod 4$.

For $r = 1$ the only restriction is $e < 3$, so that $d\chi(n,d) \equiv n\mu(d) + n^2\mu(d/2) = \mu(d)(n - n^2) = \mu(q)n(n-1) \mod 8$. Again if d is not squarefree all terms vanish. Otherwise we have $\chi(n,d) \equiv q\mu(q)n(n-1) \mod 8$ which leads to the dependencies as given above (when $n \equiv 2 \mod 4$ we have $n(n-1) \equiv n \mod 8$). $\square$

Next we perform the modulo computations for $\chi'(n,d)$, which are very similar.

Lemma 2.3.29. *Suppose n is odd, $n \neq 1$. Then $\chi'(n,d) \equiv 0 \mod 4$ if d is not a power of 2 or if $n \equiv \pm 1 \mod 8$, and $\chi'(n,d) \equiv 2 \mod 4$ if d is a power of 2 and $n \equiv \pm 3 \mod 8$. We also have*

$$\chi'(n,2) \equiv \begin{cases} 0 \mod 4 & \text{if } n \equiv 1 \mod 8, \\ 1 \mod 4 & \text{if } n \equiv 3 \mod 8, \\ 2 \mod 4 & \text{if } n \equiv 5 \mod 8, \\ 3 \mod 4 & \text{if } n \equiv 7 \mod 8. \end{cases}$$

Proof. Suppose first that d is not a power of 2, so that from Lemma 2.3.19 we are interested in the value modulo 2^{r+2} of

$$\begin{aligned}\sum_{e|d,e \text{ odd}} \mu(e)n^{d/2e} &= \sum_{e|q} \mu(\frac{q}{e})n^{2^{r-1}e} \\ &\equiv \sum_{e|q} \mu(e)n^{2^{r-1}} = 0\end{aligned}$$

with the same reasoning as in Corollary 2.3.26.

When $d = 2^r$, we are interested in $n^{2^{r-1}} - 1 \mod 2^{r+2}$ so the result follows from Corollary 2.3.25 when $r \neq 1$, and when $d = 2$ we have $\chi'(n, 2) = \frac{n-1}{2}$ and the result follows. $\square$

Lemma 2.3.30. *Suppose* $n \equiv 0 \mod 4$*, and* $d = 2^r q$ *where* q *is odd. Then* $\chi'(n, d) \equiv 2 \mod 4$ *if* $n \equiv 4 \mod 8$ *and* d *is squarefree, and otherwise* $\chi'(n, d) \equiv 0 \mod 4$.

Proof. When d is not a power of 2, the argument is identical to that of Lemma 2.3.27. When d is a power of 2, we are interested in $n^{2^{r-1}} \mod 2^{r+2}$, which is 0 unless $r = 1$ and $n \equiv 4 \mod 8$, as required. $\square$

Lemma 2.3.31. *Suppose* $n \equiv 2 \mod 4$*, and* $d = 2^r q$ *where* q *is odd. Then:*

$$\chi'(n, d) \equiv \begin{cases} \frac{n}{2}q\mu(q) \mod 4 & \textit{if } r = 1, \\ q\mu(q) \mod 4 & \textit{if } r = 2, \\ 2\mu(q) \mod 4 & \textit{if } r = 3, \\ 0 \mod 4 & \textit{if } r \geq 4. \end{cases}$$

Proof. Suppose first that $q \neq 1$. Then a similar computation to that done in Lemma 2.3.28 concludes that $\chi'(n, d) \equiv 0 \mod 4$ if $r \geq 4$. The other computations are similar to those in the corresponding case for χ, and yield the following:

- For $r = 3$, $d\chi'(n, d) \equiv n^4\mu(q) \mod 32$ so $\chi'(n, d) \equiv 2\mu(q) \mod 4$.
- For $r = 2$, $d\chi'(n, d) \equiv n^2\mu(q) \mod 16$ so $\chi'(n, d) \equiv q\mu(q) \mod 4$.
- For $r = 1$, $d\chi'(n, d) \equiv n\mu(q) \mod 8$ so $\chi'(n, d) \equiv \frac{n}{2}q\mu(q) \mod 4$.

$\square$

We summarise these computations below.

Corollary 2.3.32. *Suppose $d = 2^r q$. Then we have:*

$$\chi(n,d) - \chi'(n,d) \equiv \begin{cases} 0 \mod 4 & \text{if } n \equiv 0,1,3 \mod 4, d \neq 2, \\ 0 \mod 4 & \text{if } n \equiv 0,1 \mod 4, d = 2, \\ 2 \mod 4 & \text{if } n \equiv 3 \mod 4, d = 2, \\ 0 \mod 4 & \text{if } n \equiv 2 \mod 4, q \text{ not squarefree}, \\ 0 \mod 4 & \text{if } n \equiv 2 \mod 4, r \neq 2, q \text{ squarefree}, \\ 2 \mod 4 & \text{if } n \equiv 2 \mod 4, r = 2, q \text{ squarefree}. \end{cases}$$

$\chi(n,d)$ is even unless $n \equiv 2 \mod 4$, $r = 1,2$ and q is squarefree.

Theorem 2.3.33. *Let s be the permutation as described in Lemma 2.3.20, and let ϵ_i be as in Lemma 2.3.4. Then $\epsilon_2 = \epsilon_1$ except when $n \equiv 2 \mod 4$ and $d = 2$.*

Proof. ϵ_2 depends on the parity of $\hat{s}$, where the number of e-cycles in $\hat{s}$ is half of the number of e-cycles in s' as described in the statement of Corollary 2.3.21. In particular it suffices to compute the number of cycles of even length in s' modulo 4. Working modulo 4, for $n \not\equiv 2 \mod 4$ this is given by:

$$\begin{aligned}\sum_{e|d, e \text{ even}} X(n,e,d) &= \sum_{e|d, e \text{ even}} \chi(n,e) + 2\delta^+\chi'(n,2e) - \delta^-\chi'(n,e) \text{ (Corollary 2.3.21)} \\ &= \sum_{e|d, e \text{ even}} \chi(n,e) + 2\delta^+\chi'(n,2e) - \chi'(n,e) \\ &\equiv \sum_{e|d, e \text{ even}} \chi(n,e) - \chi'(n,e) \quad \text{since } \chi'(n,2e) \text{ is even if } e \text{ is even} \\ &\equiv \chi(n,2) - \chi'(n,2) \quad \text{by Corollary 2.3.32}\end{aligned}$$

The dependency as given in Corollary 2.3.32 is precisely the dependency given for ϵ_1 in Corollary 2.2.2.

If $n \equiv 2 \mod 4$ then we can proceed as before to get:

$$\begin{aligned}\sum_{e|d, e \text{ even}} X(n,e,d) &= \sum_{e|d, e \text{ even}} \chi(n,e) + 2\delta^+\chi'(n,2e) - \chi'(n,e) \\ &= \sum_{e|d, e \text{ even}} (\chi(n,e) - \chi'(n,e)) + 2 \sum_{e|d, e \text{ even}} \delta^+\chi'(n,2e).\end{aligned}$$

In the first sum, the summands are nonzero precisely when $e = 4t$ for t a squarefree factor of q. Notice that the number of squarefree factors of q is even unless $q = 1$, so the first sum vanishes except when $d = 2^r$, whereupon the sum is 0 if $r = 1$ and 2 if $r \geq 2$.

Similarly for the second sum, the summands are 0 unless $e = 2t$ for t a squarefree factor of q, and $\delta^+ = 1$; thus we are again counting the number of squarefree factors of q. Thus the second sum takes the same (even) value as the first sum. Hence $\epsilon_2 = 0$ whenever $n \equiv 2 \mod 4$. This agrees with Corollary 2.3.32 except when $d = 2$. □

2.4 Quasideterminants of permutation matrices in even characteristic

Recal the definition of quasideterminant, and in particular the alternate characterisation given in Lemma 1.6.30.

Lemma 2.4.1. *Let $g \in G := \mathrm{GO}_n^\epsilon(q)$ be a permutation matrix with q even, and let $c(g)$ denote the number of cycles in the permutation corresponding to g. Then the quasideterminant of g is 1 if and only if $c(g)$ is even.*

Proof. Let V be the natural module of G with basis $v_1, \ldots, v_n$. Also let σ_g denote the element of $\mathrm{Sym}(n)$ corresponding to g, and let θ be a cycle in σ_g permuting $\Lambda \subseteq \{1, \ldots, n\}$. Then the vector $w_\theta = \sum_{\lambda \in \Lambda} v_\lambda$ is such that $w_\theta g = w_\theta$, so that $w_\theta \in \mathrm{Null}(I_n - g)$, and clearly choices of disjoint cycles θ give rise to linearly independent vectors w_θ. In particular we have $c(g) \leq \dim(\mathrm{Null}(I_n - g))$.

Conversely, suppose $w = \sum_{i=1}^{n} \mu_i v_i \in \mathrm{Null}(I_n - g)$. Then in particular we must have $wg = w$, so that $\mu_{i^g} = \mu_i$. Thus coefficients of w are constant on orbits of σ_g, so that w is in the $\mathbb{F}_q$-span of the vectors w_θ described in the previous paragraph and so $c(g) = \dim(\mathrm{Null}(I_n - g))$. □

Thus we can determine the quasideterminant of the matrix entirely from the cycle structure of the corresponding permutation.

From now on, let $G = \Omega_n^\epsilon(q^d)$ and let g_σ denote the permutation inducing the field automorphism $\sigma_{\hat{G}}$ of the tensor field group $\hat{G} = G \otimes G^\sigma \otimes \cdots \otimes G^{\sigma^{d-1}}$. From the above, we can determine the quasideterminant of g_σ from the parity of $c(g)$, which in the notation of Lemma 2.3.19 is $\psi(n, d)$.

Lemma 2.4.2. *If d is odd then $\psi(n, d)$ is even if n is even, and odd if n is odd.*

Proof. If d is odd then σ must be a product of even cycles (i.e. cycles of odd length) since all factors of d must also be odd. In particular there can be no self-dual cycles in σ, so that every cycle has a corresponding dual cycle, with the exception of the fixed point $\frac{n^d+1}{2}$ if n is odd. Thus we have an even number of cycles if n is even (so there is no self-dual fixed point), and an odd number of cycles if n is odd (every cycle is paired with its dual cycle, plus the self-dual fixed point). □

Recall from the proof of Lemma 2.3.19 that we can write $\psi(n,d) = \sum_{e|d} \chi(n,e)$.

Lemma 2.4.3. *If d is odd and $d > 1$, then $\chi(n,d)$ is even. If $d = 1$ then $\chi(n,1) = n$ and so $\chi(n,1)$ is even if n is even and odd if n is odd.*

Proof. Proceed by induction on d. When $d = 1$, $\chi(n,1) = \psi(n,1) = n$ follows from the definition. Now suppose that $d > 1$ and the result holds for all odd e with $1 < e < d$. In particular it holds for all nontrivial factors of d. Then we have from Lemma 2.4.2 that $\psi(n,d) \equiv n \mod 2$, whilst by induction

$$\sum_{e|d} \chi(n,e) \equiv \chi(n,1) + \chi(n,d) \equiv n + \chi(n,d) \mod 2.$$

Since $\psi(n,d) = \sum_{e|d} \chi(n,e)$, it follows that $\chi(n,d) \equiv 0 \mod 2$. $\square$

Lemma 2.4.4. *If d is even and n is odd, then $\psi(n,d)$ is even if $n \equiv 3 \mod 4$ and odd if $n \equiv 1 \mod 4$.*

Proof. Lemma 2.4.3 tells us that when e is odd, $\chi(n,e) \equiv 1 \mod 2$ if and only if $e = 1$. From Corollary 2.3.26 we see that when e is even $\chi(n,d) \equiv 1 \mod 2$ if and only if $d = 2$ and $n \equiv 3 \mod 4$. Thus we have $\psi(n,d) = \sum_{e|d} \chi(n,e) \equiv 1 + \chi(n,2) \mod 2$. Hence the result follows. $\square$

Lemma 2.4.5. *If d is even and $n \equiv 0 \mod 4$, then $\psi(n,d)$ is even.*

Proof. Lemma 2.3.27 and Lemma 2.4.3 tell us that $\chi(n,e)$ is even for every choice of e; hence via $\psi(n,d) = \sum_{e|d} \chi(n,e)$ so is $\psi(n,d)$. $\square$

Lemma 2.4.6. *If d is even and $n \equiv 2 \mod 4$, then $\psi(n,d)$ is even unless $d = 2$, when $\psi(n,2)$ is odd.*

Proof. From Lemma 2.3.28 and Lemma 2.4.3, we have that when $n \equiv 2 \mod 4$, for $e|d$ the quantity $\chi(n,e)$ is odd if and only if $e = 2^r t$ with $r = 1,2$ and t odd and squarefree. Thus the parity of $\psi(n,d)$ depends on the parity of the size of the set $T := \{2^r t : r = 1,2, t \text{ odd and squarefree }, 2^r t|d\}$. If $4|d$, this set clearly has even size, so suppose $d = 2s$ where s is odd. Let $p_1, \dots, p_k$ be a list of distinct primes that divide s; then every element of T is of the form $2p_1^{\delta_1} \dots p_k^{\delta_k}$ where $\delta_i \in \{0,1\}$. Hence there are an even number of these elements unless $k = 0$; i.e. unless $d = 2$. In this case T contains only one element, so that this is the only situation where $\psi(n,d)$ is odd. $\square$

We collate the above results into a single theorem for future reference.

Theorem 2.4.7. *Suppose d is even. Then $\psi(n, d)$ is even if $n \equiv 0, 3 \mod 4$ or if $n \equiv 2 \mod 4$ and $d > 2$; otherwise $\psi(n, d)$ is odd (i.e. when $n \equiv 1 \mod 4$, or $n \equiv 2 \mod 4$ and $d = 2$).*

2.5 Applications

Lemma 2.5.1. *Let $G = \mathrm{GO}_n^\delta(q^d)$ where q is odd and d is even preserving the form $\mathrm{antidiag}(1, \dots, 1)$ (so $\delta \in \{+, \circ\}$) with natural module V. Let $\hat{G}$ be the tensor field group of G, and $H < \mathrm{GO}_{n^d}^\epsilon(q)$ be a rewritten tensor field group of G. Let the matrix g_σ induce the σ automorphism on $\hat{G}$ by conjugation. Then:*

(i) If $n \equiv 0 \mod 4$ then $\epsilon = +$, $\det g_\sigma = 1$ and the spinor norm of g_σ with respect to the form $\mathrm{antidiag}(1, \dots, 1)$ over $\mathbb{F}_q$ is 1 for every q.

(ii) If $n \equiv 1 \mod 4$ then $\epsilon = \circ$, $\det g_\sigma = 1$ and the spinor norm of g_σ with respect to the form $\mathrm{antidiag}(1, \dots, 1)$ over $\mathbb{F}_q$ is 1 for every q.

(iii) If $n \equiv 2 \mod 4$ then if $d \neq 2$ then $\epsilon = +$, $\det g_\sigma = 1$ and the spinor norm of g_σ with respect to the form $\mathrm{antidiag}(1, \dots, 1)$ over $\mathbb{F}_q$ is 1 for every q. When $d = 2$, $\epsilon = -$ and $\det g = -1$.

(iv) If $n \equiv 3 \mod 4$ then $\epsilon = \circ$, $\det g_\sigma = -1$ and the spinor norm of $-g_\sigma$ with respect to the form $\mathrm{antidiag}(1, \dots, 1)$ over $\mathbb{F}_q$ is 1 if $q \equiv 1 \mod 4$ and -1 otherwise.

Proof. Lemma 2.3.1 gives the conditions on ϵ. The conditions on g_σ for all except (iv) follow from from Theorem 2.3.33, Corollary 2.2.2 and Lemma 2.3.4.

In the case when $n \equiv 3 \mod 4$, we are interested in the spinor norm of $-g$. It follows easily from Lemma 1.6.30 that the spinor norm of $-I_n$ is 1 if and only if $\det(\mathrm{antidiag}(2, \dots, 2)) = (-1)^{(n-1)/2} 2^n$ is a square in $\mathbb{F}_q$. Since $n \equiv 3 \mod 4$, it follows that $(-1)^{(n-1)/2} = -1$, and so the spinor norm of $-I_n$ depends on whether -2 is a square in $\mathbb{F}_q$; i.e. it has spinor norm -1 if $q \equiv 5, 7 \mod 8$ and otherwise it has spinor norm 1. Note that by the previously stated results, g_σ has spinor norm 1 if $q \equiv \pm 1 \mod 8$ and -1 otherwise. Hence, since the spinor norm is a homomorphism, the conditions follow. $\square$

Lemma 2.5.2. *Let $G := \mathrm{GO}_n^\delta(q^d)$ with q even and $d \geq 2$. Let $\hat{G}$ be the tensor field group of G, and $H < \mathrm{GO}_{n^d}^\epsilon(q)$ be a rewritten tensor field group of G. Let $g_\sigma \in \mathrm{SO}_{n^d}^\epsilon(q)$ induce the $\sigma_{\hat{G}}$ automorphism on $\hat{G}$ by conjugation. Then:*

(i) The quasideterminant of g_σ is 1 if $n \equiv 0, 3 \mod 4$ or $n \equiv 2 \mod 4$ and $d > 2$.

(ii) The quasideterminant of g_σ is -1 if $n \equiv 1 \mod 4$ or $n \equiv 2 \mod 4$ and $d = 2$.

Proof. The quasideterminant of g_σ depends on the parity of $\psi(n, d)$, which is given by Theorem 2.4.7. □

Lemma 2.5.3. *Let $n > 2$ and d both be even, and suppose either $d > 2$ or $n \equiv 0 \mod 4$. Let $G = \mathrm{Sp}_n(q^d)$ with q odd and G preserving the form $f = \mathrm{antidiag}(1, \ldots, 1, -1, \ldots, -1)$, and let V be the natural module of G with basis $e_1, \ldots, e_n$. Let $\hat{G}$ be the tensor field group of G over $\mathbb{F}_q$, and $H < \mathrm{GO}^{\epsilon}_{n^d}(q)$ be a rewritten tensor field group of G over $\mathbb{F}_q$. Let g_σ be a matrix inducing the σ automorphism on $\hat{G}$ by conjugation. Then $\epsilon = +$, $\det g_\sigma = 1$ and the spinor norm of g_σ with respect to the form given by the d-fold tensor product of f over $\mathbb{F}_q$ is 1 for every q.*

Proof. From the conditions on n and d, we have by Lemma 2.3.1 that H preserves a form of orthogonal plus type. The group $\hat{G}$ preserves a symmetric antidiagonal form whose non-zero entries lie in $\{1, -1\}$. We can decompose the permutation matrix g_σ into a direct sum of two permutation matrices g_e and g_o preserving forms $\mathrm{antidiag}(1, \ldots, 1)$ and $\mathrm{antidiag}(-1, \ldots, -1)$ respectively. In the notation of Lemma 2.3.4, let ϵ_i^e, ϵ_i^o and ϵ_i denote the values of ϵ_i on g_e, g_o and g_σ respectively, so that $\epsilon_i^e + \epsilon_i^o = \epsilon_i$, working modulo 2. From Lemma 2.3.4 we have the spinor norm of g_e depends on $(-2)^{\epsilon_1^e}(-1)^{\epsilon_2^e}$, and from Corollary 2.3.6 the spinor norm of g_o depends on $(-2)^{\epsilon_1^o}(-1)^{\epsilon_2^o}$. Hence the spinor norm of g_σ depends on $(-2)^{\epsilon_1^o} 2^{\epsilon_1^e}(-1)^{\epsilon_2^o + \epsilon_2^e} = 2^{\epsilon_1}(-1)^{\epsilon_2 + \epsilon_1^o}$.

The matrix g_o acts on the set X from Lemma 2.2.4. Also from Lemma 2.2.4 we see that the embedding is even in all the cases we are interested in, so that g_o has determinant 1, and so $\epsilon_1^o = 0$. From Theorem 2.3.33 we know $\epsilon_1 = \epsilon_2$, and from Corollary 2.2.2 we know $\epsilon_1 = 0$, so the result follows. □

Remark 2.5.4. If Conjecture 2.3.3 holds, then the conclusions of Lemmas 2.5.1, 2.5.2 and 2.5.3 will also hold for g_σ^c, where c is the matrix conjugating $\hat{G}$ to H.

Chapter 3

$\mathcal{S}_1$-candidates

3.1 Introduction

In this chapter, we aim to classify the $\mathcal{S}_1$-candidates in dimensions 16 and 17, using similar methods to those in [8]. Section 3.2 offers a brief introduction to the methods used. Sections 3.3 and 3.4 deal with all $\mathcal{S}_1$-candidates in dimensions 17 and 16 respectively - we will consider the 17-dimensional case first as this is more straightforward and allows us to give the proofs in more detail.

A number of proofs use computations in Magma [5]. Some computations are straightforward and for these no code is provided to check these claims; other computations reference files which can be found at
`https://github.com/danielrogerswarwick/thesis`.

3.2 The theory of $\mathcal{S}_1$-candidates

In this section, we provide a summary of the methods we will use to determine the $\mathcal{S}_1$-maximal candidates. For more detail regarding these methods, see [8, Chapter 4]. There is some overlap between work here and results in [34], which gives conjugacy classes and normalisers of the candidate subgroups as subgroups of $\mathrm{GL}_n(q)$ for $13 \leq n \leq 27$; we extend this work by also providing class stabilisers, and also by finding the normalisers inside classical groups other than $\mathrm{GL}_n(q)$ when the representation preserves a nonzero form.

3.2.1 Candidates

The first task is to produce a list of all quasisimple groups with a 16- or 17-dimensional absolutely irreducible representation in non-defining characteristic.

A list of all such groups in dimension up to 250 has been produced in [23] and [24]. In Sections 3.3.1 and 3.4.1 we summarise the relevant information, alongside other useful information about the representation, described in more detail in Remark 3.3.2.

A number of the Magma calculations require constructions of the relevant representations; these are usually obtained from libraries either in Magma or in [60] and [61].

In the tables of candidates we will often record several equivalence classes of representations in the same row, following the convention in [8, p. 159] which we describe below.

Definition 3.2.1. Let ρ_1 and ρ_2 denote absolutely irreducible representations of G over $\mathbb{F}_q$. Then ρ_1 and ρ_2 are *algebraically conjugate* if either:

(i) There exist representations ρ_1' and ρ_2' over $\mathbb{C}$ such that ρ_i is the p-modular reduction of ρ_i', and there exists a field automorphism σ of $\mathbb{C}$ such that $(\rho_1')^\sigma = (\rho_2')$; or

(ii) Corresponding representations ρ_1' and ρ_2' over $\mathbb{C}$ do not exist, and there exists a field automorphism σ of $\mathbb{F}_q$ such that $\rho_1^\sigma = \rho_2$.

Definition 3.2.2. Let ρ_1 and ρ_2 denote absolutely irreducible representations of the same group G. Then ρ_1 and ρ_2 are *weakly equivalent* if there exists an automorphism $\alpha \in \mathrm{Aut}(G)$ such that ${}^\alpha\rho_1$ is algebraically conjugate to ρ_2 or ρ_2^*.

In other words, the weak equivalence class of a representation ρ_1 is its closure under the actions of quasi-equivalence, algebraic conjugation and duality. It is usually not hard to determine whether two representations are weakly equivalent by looking at the character table.

Example 3.2.3. We consider 17-dimensional representations of $G = \mathrm{L}_2(16)$ in non-dividing characteristic, using the character table and notation as given in [12]. Looking at the character table in characteristic 0, we see seven 17-dimensional characters, which we denote $\chi_{11}, \ldots, \chi_{17}$. Note that all the operations in the definition of weak equivalence preserve character values which lie in $\mathbb{Z}$, meaning that we have at least three weak equivalence classes here (considering for instance elements of order 2); namely $\{\chi_{11}\}$, $\{\chi_{12}, \chi_{13}\}$ and $\{\chi_{14}, \chi_{15}, \chi_{16}, \chi_{17}\}$. Hence χ_{11} is weakly equivalent only to itself. We can obtain χ_{13} from χ_{12} by applying the field automorphism of $\mathbb{C}$ which interchanges the two roots of $X^2 + X - 1$, the minimal polynomial of the algebraic number b_5; hence χ_{12} and χ_{13} form a single weak equivalence class. The field automorphism of G also interchanges these two representations. Similarly, a field automorphism of $\mathbb{C}$ interchanges $\chi_{14}, \ldots, \chi_{17}$; these are also cyclically permuted by the outer automorphism of order 4 of $\mathrm{L}_2(16)$ inducing the field automorphism on G. Hence these four characters also form a weak equivalence class for suitable finite fields $\mathbb{F}_q$. Note that all these characters are self-dual.

Each line in the table of candidates consists of a single weak equivalence class.

3.2.2 Forms and fields

For each weak equivalence class of representations, we now seek to determine the smallest classical group which contains the image of a representative of the weak equivalence class.

Lemma 3.2.4. *Suppose ρ is an n-dimensional representation of a nonabelian simple group G in characteristic p. Then $G\rho < \mathrm{SL}_n(F)$, where F is the smallest field of characteristic p such that the p-modular reduction of every irrationality in the character ring of ρ can be realised over F.*

Proof. It follows from Lemma 1.7.19 that $G\rho < \mathrm{GL}_n(F)$. Since G is nonabelian simple, $G = [G, G]$, and so $G < [\mathrm{GL}_n(F), \mathrm{GL}_n(F)]$. When $(n, q) \neq (2, 2)$ it follows from [46, Theorem 8.20] that $[\mathrm{GL}_n(F), \mathrm{GL}_n(F)] = \mathrm{SL}_n(F)$ and the result follows, whilst $\mathrm{GL}_2(2)$ contains no nonabelian simple subgroup. □

As part of the definition of Class $\mathcal{S}$, we require that G^∞ is written over the smallest possible field in the given characteristic, and that it preserves no additional forms.

Definition 3.2.5. Let χ be a character of a group G. Then the *Schur indicator* (or *Frobenius-Schur indicator*), denoted $v_2(\chi)$, is defined by:

$$v_2(\chi) = \frac{1}{|G|} \sum_{g \in G} g^2\chi.$$

Lemma 3.2.6. *[30, Theorem 4.5] $v_2(\chi) \in \{-1, 0, 1\}$.*

Lemma 3.2.7 (Frobenius-Schur)**.** *[14, Theorem 73.13] Let ρ be a representation of a group G, with corresponding character χ.*

(i) The group $G\rho$ preserves a quadratic form if and only if $v_2(\chi) = 1$.

(ii) The group $G\rho$ preserves a symplectic form if and only if $v_2(\chi) = -1$.

(iii) If $v_2(\chi) = 0$, then the group $G\rho$ either preserves a unitary form, or the zero form and no other classical form.

Thus from the Schur indicator of ρ we can generally determine the classical form preserved by $G\rho$. We use the notation $+, -$ and $\circ$ to denote representations with Schur indicator $+1, -1$ and 0 respectively.

For a representation with Schur indicator $-$, $G\rho < \mathrm{Sp}_n(q)$. In particular, n must be even.

For a representation with Schur indicator $+$, $G\rho < \Omega_n^\epsilon(q)$ if G is perfect (due to an argument similar to Lemma 3.2.4). If n is odd then $\epsilon = \circ$, whereas if n is even we will need to determine the sign of ϵ. This is usually done by explicitly constructing the form in question and finding its determinant.

For a representation with Schur indicator $\circ$, we need to decide whether $G\rho$ preserves a unitary form or not. For this the following result is sufficient.

Lemma 3.2.8. *[8, Lemma 4.4.1 and Corollary 4.4.2] Let $\widehat{\rho}$ denote an absolutely irreducible representation of a group G with Schur indicator $\circ$ over $\mathbb{C}$ and corresponding character $\widehat{\chi}$. Suppose that the character ring of $\widehat{\rho}$ is generated by irrationalities $\widehat{a_1}, \ldots, \widehat{a_n}$. Suppose that $G\widehat{\rho}$ has a p-modular reduction over $\mathbb{F}_{q^2}$; denote the p-modular reduction of $\widehat{\rho}$, $\widehat{\chi}$ and $\widehat{a_i}$ by ρ, χ and a_i respectively. Then $G\rho$ preserves a unitary form if and only if the automorphism $\sigma : \mathbb{F}_{q^2} \to \mathbb{F}_{q^2}, a \mapsto a^q$ acts on χ as complex conjugation.*

Equivalently, $G\rho$ preserves a unitary form if and only if

$$\widehat{a_i} \in \mathbb{R} \iff a_i \in \mathbb{F}_q.$$

3.2.3 Actions of automorphisms of the classical group

The procedure up until this point is sufficient to determine G^∞ for all $\mathcal{S}_1$-candidate subgroups G of any quasisimple classical group Ω. However, to determine the structure of G we need to compute the normaliser $N_\Omega(G)$. Further, if we are interested in determining maximal subgroups of extensions T of Ω by automorphisms, we also need to determine $N_T(G)$. Recall that T is such that $\overline{\Omega} < \overline{T} < \mathrm{Aut}(\overline{\Omega})$.

Elements in the normaliser $N_H(G)$ fall into one of three classes, following Lemma 1.7.11:

- Elements of G, which act via conjugation as inner automorphisms of G.
- Central elements in T, which commute with elements of G.
- Elements $t \in T$ which induce an outer automorphism of G.

Note that when considering an extension of $G\rho$ by a subgroup $H < \mathrm{Aut}(G)$, we have that $((G\rho).H)^\infty = G\rho$; thus recalling the definition of class $\mathcal{S}$, having determined the minimal field of definition and forms preserved by $G\rho$ from the previous section, for $(G\rho).H$ to be a maximal subgroup of a classical group the matrices which induce the automorphisms in H must also lie over the minimal field and preserve the same form as $G\rho$. We will see later in this section why such conditions are necessary for maximality.

Definition 3.2.9.

- If $N_T(G) = G$, then we say that G is *self-normalising.*
- If $N_T(G) < ZG$, where $Z = Z(T)$ is the group of scalar matrices in T, then we say that G is *scalar normalising.*

Lemma 3.2.10. *[8, Lemma 4.4.3] Let G be a quasisimple group, and ρ_1 a faithful absolutely irreducible representation of G. Let $\{\rho_1, \ldots, \rho_r\}$ be a set of representatives of the equivalence classes of representations that are weakly equivalent to ρ_1, and let C be the corresponding conformal group of the smallest classical group Ω that contains $G\rho_1$. Then:*

(i) There is a natural bijection between the orbits of $\mathrm{Out}(G)$ *on $\{\rho_1, \ldots, \rho_r\}$ and the conjugacy classes into which C partitions $\{G\rho_1, \ldots, G\rho_r\}$.*

(ii) Each C-class of subgroups splits into $|C : N_C(G)\Omega|$ classes in Ω.

(iii) The outer automorphisms of G which are induced by elements of $N_C(G)$ are precisely those that stabilise ρ_1.

Hence, given a representation ρ of a group G, we consider the outer automorphisms of G separately, depending on whether they stabilise ρ or not.

If the automorphism $\alpha \in \mathrm{Out}(G)$ stabilises ρ then there exists a matrix $g_\alpha \in C$, for C the conformal group of Ω, such that conjugation by g_α induces α on $G\rho$. We can then perform calculations on g_α to determine whether it preserves the form, and to find its determinant and (in the orthogonal case) spinor norm. There are two possibilities here:

(1) It may be the case that $g_\alpha \in \Omega$. In this case, we have that $g_\alpha \in N_\Omega(G)$. In other words, we can realise the group $G.\alpha$ as a subgroup of Ω.

(2) Otherwise, $g_\alpha \notin \Omega$, and hence α is induced by some outer automorphism of Ω. Depending on the shape of Ω, we may need to perfom additional calculations on g_α to determine precisely which outer automorphism $\beta \in \mathrm{Out}(\Omega)$ this is. Then, we have that $g_\alpha \in N_S(G)$ where S is such that $\Omega < S < \mathrm{Aut}(\Omega)$ and $\beta \in S$.

For the automorphisms $\alpha \in \mathrm{Out}(G)$ which do not stabilise the representation ρ, we have that ρ and ${}^{\alpha}\rho$ are distinct elements of the weak equivalence class of ρ. There may exist an automorphism $\beta \in \mathrm{Out}(\Omega)$ such that ${}^{\alpha}\rho = \rho^{\beta}$; in this case, it follows from Lemma 1.7.13 that the matrix α is induced by the automorphism βc_g of Ω for some matrix $g \in C$, where c_g denotes the automorphism of Ω obtained by conjugation by g. Again, we may need to compute whether g can be rescaled to preserve the form, as well as its determinant and spinor norm, in order to determine which automorphism of Ω is induced by βc_g. Recall that as a consequence of Lemma 3.2.10 β cannot be induced by an element of the conformal group of the classical group, and hence must be either a field automorphism, or a graph automorphism with $\Omega = \mathrm{SL}_n^{\pm}(q)$ (or an exceptional graph automorphism, but we will not need to consider these in this thesis).

Note that we will only need to determine class stabilisers up to conjugacy in $\mathrm{Out}(\Omega)$.

The following two lemmas justify why, having embedded a quasisimple group G inside an n-dimensional classical group Ω, we do not need to consider embeddings of $G.\alpha$ inside n-dimensional classical groups other than Ω.

Lemma 3.2.11. *Let ρ be an absolutely irreducible representation of a quasisimple group G in characteristic p. Let $\alpha \in \mathrm{Out}(G)$ stabilise ρ. Suppose that $G\rho$ and $(G\rho).\alpha$ are both realisable as matrices over $\mathbb{F}_q$ (for q as small as possible). Suppose additionally that $G\rho < \Omega$ and $(G\rho).\alpha < \Omega'$, where Ω and Ω' are classical groups over $\mathbb{F}_q$ preserving differing forms. Then $(G\rho).\alpha$ is not a maximal subgroup of Ω'.*

Proof. Note that since $G\rho < (G\rho).\alpha$, it follows that $G\rho$ preserves the forms of both Ω and Ω'. In particular, it follows from Lemma 1.7.10 that Ω and Ω' cannot preserve distinct bilinear or unitary forms. Further, by definition of the Schur indicator if $G\rho$ has Schur indicator $\circ$ then $(G\rho).\alpha$ cannot have Schur indicator $+$ or $-$, as otherwise $G\rho$ would also preserve a bilinear form. If $G\rho$ preserves a bilinear form and $(G\rho).\alpha$ preserves a unitary form, then by Lemma 1.7.10 $G\rho$ is expressible over a smaller field than $\mathbb{F}_q$, contradicting the minimal choice of q. Also, if $G\rho$ preserves no form then it must follow that $(G\rho).\alpha$ also preserves no form.

Hence the only remaining possibility is that $(G\rho).\alpha$ embeds in some classical group Ω, and $\Omega' = \mathrm{GL}_n(q)$. In this case, it follows from Lemma 3.2.10 that there exists $g \in C$ such that g normalises and induces α on $G\rho$, where C is the conformal group of Ω. By assumption we also have $h \in \mathrm{GL}_n(q)$ such that h induces α on $G\rho$. Hence since ρ is absolutely irreducible it follows that gh^{-1} centralises $G\rho$ and thus from Lemma 1.7.3 there exists a scalar $\lambda \in \mathbb{F}_q$ such that $h = \lambda g$. Since

g lies in C, it follows that $\lambda g \in C$ and hence we have the chain of subgroups $(G\rho).\alpha < C < \mathrm{GL}_n(q)$. Hence $(G\rho).\alpha$ is not maximal in $\mathrm{GL}_n(q)$. □

Lemma 3.2.12. *Let ρ be an absolutely irreducible representation of a quasisimple group G in characteristic p. Let Ω_f be a family of classical groups over $\mathbb{F}_{q^f}$, with q a power of p, and q and Ω_f chosen such that $G\rho < \Omega_1$. Let $\alpha \in \mathrm{Out}(G)$ stabilise ρ, and suppose that $(G\rho).\alpha < \Omega_e$, with e as small as possible. Then if $e > 1$, $(G\rho).\alpha$ is not a maximal subgroup of Ω_e.*

Proof. Let C_f denote the conformal group of Ω_f. Recall from Lemma 3.2.10 that there exists an element $g \in C_1$ which normalises and induces α on $G\rho$. If $e > 1$, then by definition of e we have $g \notin \Omega_1$. Also by assumption, there exists a matrix $h \in \Omega_e$ which also normalises and induces α on $G\rho$. Since ρ is absolutely irreducible, it follows that gh^{-1} centralises $G\rho$ and thus from Lemma 1.7.3 there exists a scalar $\lambda \in \mathbb{F}_{q^e}$ such that $h = \lambda g$. Thus, we have that $h \in \Omega_e \cap ZC_1$, where $Z = Z(\mathrm{GL}_n(q^e))$. Hence we have $(G\rho).\alpha = (G\rho).\langle h\rangle < \Omega_e \cap ZC_1 < \Omega_e$, where all inclusions are strict; hence $(G\rho).\alpha$ is not a maximal subgroup of Ω_e. □

Hence, having determined the minimal field of realisation of a representation ρ of G, it follows that all extensions of $G\rho$ by automorphisms must also lie over the same field.

In the following subsections, we will describe the methods we use for determining the class stabilisers for each of the classical groups in turn. Throughout these sections, let G be a quasisimple group.

Linear case

In this section, let $\Omega = \mathrm{SL}_n(q)$. Recall the construction of the automorphism group of Ω in Section 1.6.4. In all the cases we will need to consider, the field automorphism is trivial, and the following results will be sufficient for our purposes.

Lemma 3.2.13. *[8, Lemma 4.6.1] Let $\rho : G \to \mathrm{SL}_n(q)$ be a representation. Suppose there exists $\alpha \in \mathrm{Aut}(G)$ such that ${}^{\alpha}\rho = \rho^{\gamma}$, and that $d = (q-1, n)$ is odd. Then the stabiliser of the class of $G\rho$ in Ω contains a conjugate of γ in $\mathrm{Out}(\Omega)$.*

Lemma 3.2.14. *[8, Lemma 4.6.2] Let $\rho : G \to \mathrm{SL}_n(q)$ be a representation and suppose $d = (q-1, n)$ is even. Let $\beta \in \langle\phi, \gamma\rangle$, and suppose β is such that $\beta\delta^i$ and $\beta\delta^j$ are conjugate by a power of δ when $i - j$ is even. Suppose also that there exists $\alpha \in \mathrm{Aut}(G)$ such that ${}^{\alpha}\rho = \rho^{\beta}$, and $L \in \mathrm{GL}_n(q)$ such that $L^{-1}(x\rho)^{\beta}L = (x^{\alpha})\rho$ for all $x \in G$. Then the stabiliser of the class of $G\rho$ in Ω contains a conjugate of β*

in $\mathrm{Out}(\Omega)$ *if* $\det L$ *is a square in* $\mathbb{F}_q^*$, *and the stabiliser contains a conjugate of* $\beta\delta$ *otherwise.*

Note that $\gamma\delta^i$ and $\gamma\delta^j$ are conjugate by $\delta^{\frac{i-j}{2}}$ when $i-j$ is even, so in particular we can apply Lemma 3.2.14 with $\beta = \gamma$.

Unitary case

In this section, let $\Omega = \mathrm{SU}_n(q, I_n)$. This situation is complicated by the fact that the structure of $\mathrm{Out}(\mathrm{SU}_n(q, B))$ depends on B in general - see [7, Section 3] for details. Computations may construct groups preserving unitary forms other than I_n, and so the computations of the class stabiliser will often require transforming B first.

Recall the construction of the automorphism group of Ω with respect to the standard form I_n in Section 1.6.5.

Lemma 3.2.15. *[8, Lemma 4.6.1] Let* $\rho : G \to \mathrm{SU}_n(q, B)$ *be a representation. Then there exists* $A \in \mathrm{GL}_n(q^2)$ *such that* $AA^{\sigma T} = B$, *and* $(G\rho)^A < \mathrm{SU}_n(q, I_n)$. *Suppose there exists* $\alpha \in \mathrm{Aut}(G)$ *such that* ${}^{\alpha}\rho = \rho^{\gamma}$, *and that* $d = (q+1, n)$ *is odd. Then the stabiliser of the class of* $(G\rho)^A$ *in* Ω *contains a conjugate of* γ *in* $\mathrm{Out}(\Omega)$.

Lemma 3.2.16. *[8, Lemma 4.6.4 and Lemma 4.6.5] Let* $\rho : G \to \mathrm{SU}_n(q, B)$ *be an absolutely irreducible representation, and suppose* $d = (q+1, n)$ *is even. Let* β *be one of* γ *or* ϕ. *Suppose there exists* $\alpha \in \mathrm{Aut}(G)$ *such that* ${}^{\alpha}\rho$ *and* ρ^{β} *are equivalent. Then:*

(i) There exists $A \in \mathrm{GL}_n(q^2)$ *such that* $AA^{\sigma T} = B$, *and* $(G\rho)^A < \mathrm{SU}_n(q, I_n)$.

(ii) There exists $L \in \mathrm{GL}_n(q^2)$ *such that* $L^{-1}(x\rho)^{\beta}L = (x^{\alpha})\rho$. *Further,* $LBL^{\sigma T} = \lambda B^{\beta}$ *for some* $\lambda \in \mathbb{F}_q^*$, *and there exists* $\kappa \in \mathbb{F}_{q^2}^*$ *such that* $\kappa^2 = \det L$.

(iii) The stabiliser of the class of $(G\rho)^A$ *in* Ω *contains a conjugate of* β *when* $\eta = 1$ *and a conjugate of* $\beta\delta$ *when* $\eta = -1$, *where* η *is as follows:*

$$\eta := \begin{cases} \lambda^{-n/2}\kappa^{1+q}\det B & \text{if } \beta = \gamma, \\ \lambda^{-n/2}\kappa^{1+q}(\det B)^{(1-p)/2} & \text{if } \beta = \phi. \end{cases}$$

A number of the representations we will consider occur as p-modular reductions of representations in characteristic 0 over some number field. For these representations, it is often possible to determine the action of the graph automorphism γ without extensive computation.

Let $\widehat{\rho}$ denote the representation of a group G over a number field F (namely the *character field* of $\widehat{\rho}$, the number field generated by the character ring of $\widehat{\rho}$), such that $G\widehat{\rho}$ preserves a form $\widehat{B}$. Suppose that there exists $\alpha \in \mathrm{Out}(G)$ such that ${}^{\alpha}\widehat{\rho}$ is equivalent to $\widehat{\rho}^{\,\gamma}$, where γ denotes the inverse transpose automorphism. Then similarly to before we can find a matrix $\widehat{L}$ such that $\widehat{L}^{-1}(x\widehat{\rho})^{\gamma}\widehat{L} = (x^{\alpha})\widehat{\rho}$. We aim to reduce all of these objects modulo p where possible.

Definition 3.2.17. In the above context, suppose that the matrices $x\widehat{\rho}$ (for all $x \in G$), $\hat{L}$, $\hat{B}$ and their inverses can be realised over a subring of $R\left[\frac{1}{p_1}, \ldots, \frac{1}{p_m}\right]$ (where R denotes the character ring of $\widehat{\rho}$), for a finite list of primes $p_1, \ldots, p_m$, with this list as small as possible. Then we say that the p_i are *exceptional primes.*

We cannot perform p-modular reduction with respect to the exceptional primes, and we will need to perform separate computations for this finite list of exceptions. Otherwise, we can reduce all of these quantities modulo p, and denote the corresponding p-modular reduction using the same symbol with the hat removed; so for example the p-modular reduction of $\widehat{\rho}$ is ρ.

Proposition 3.2.18. *[8, Proposition 4.6.6] Suppose that G has an absolutely irreducible representation ρ with image in $\mathrm{SU}_n(q, B)$ that arises as the p-modular reduction of a characteristic 0 representation $\widehat{\rho}$ over the character field $F \subset \mathbb{C}$ of $\widehat{\rho}$, whose image preserves a unitary form $\widehat{B}$. Suppose that there exists $\alpha \in \mathrm{Out}(G)$ and $\widehat{L} \in \mathrm{GL}_n(F)$ such that $\widehat{L}^{-1}(x\widehat{\rho})^{\gamma}\widehat{L} = (x^{\alpha})\widehat{\rho}$.*

Let p be a prime which is not an exceptional prime. Suppose that $\det \widehat{L}$ factorises in $R\left[\frac{1}{p_1}, \ldots, \frac{1}{p_m}\right]$ as $\widehat{\nu}^2\widehat{\zeta}$ with $\widehat{\zeta} \in \mathbb{R}$. Let ζ be the p-modular reduction of $\widehat{\zeta}$ and define $\epsilon \in \{1, -1\}$ by $\epsilon = \begin{cases} 1 & \text{if } \sqrt{\zeta} \in \mathbb{F}_q^*, \\ -1 & \text{if } \sqrt{\zeta} \notin \mathbb{F}_q^*. \end{cases}$

Then there exists $A \in \mathrm{GL}_n(q^2)$ such that $(G\rho)^A < \mathrm{SU}_n(q, I_n)$, and the stabiliser of the class of $(G\rho)^A$ contains a conjugate of $\begin{cases} \gamma & \text{if } \epsilon\,\mathrm{sgn}(\widehat{\zeta}) = 1, \\ \gamma\delta & \text{if } \epsilon\,\mathrm{sgn}(\widehat{\zeta}) = -1, \end{cases}$ *where* sgn *denotes whether the real number is positive or negative.*

Orthogonal case - odd dimension

In this section, let $\Omega = \Omega_n^{\circ}(q)$. Recall from Section 1.6.7 that $\mathrm{Out}(\Omega) = \langle \phi, \delta | \phi^e = \delta^2 = [\delta, \phi] = 1\rangle \cong C_e \times C_2$ where $q = p^e$.

In some computations, it will be useful to know which elements of $\mathrm{Out}(\Omega)$ have certain orders. The following lemma performs this task for involutions.

Lemma 3.2.19. *[8, Lemma 4.9.40] Let g be an element of order 2 in $\mathrm{Aut}(\mathrm{O}_n^\circ(q))$, with $q = p^e$. Then*

$$g \in \mathrm{O}_n^\circ(q).\langle\delta\rangle \cup \mathrm{O}_n^\circ(q).\langle\phi\rangle.$$

In dimension 17 we will need a similar result for elements of order 4:

Lemma 3.2.20. *Let g be an element of order 4 in $\mathrm{Aut}(\mathrm{O}_n^\circ(q))$, with $q = p^e$. Then*

$$g \in \mathrm{O}_n^\circ(q).\langle\delta\rangle \cup \mathrm{O}_n^\circ(q).\langle\phi^{\frac{e}{(4,e)}}\rangle \cup \mathrm{O}_n^\circ(q).\langle\delta\phi^{\frac{e}{(2,e)}}\rangle$$

Proof. We use the isomorphism $\mathrm{Aut}(\mathrm{O}_n^\circ(q)) = \mathrm{O}_n^\circ(q).\langle\phi,\delta\rangle \cong \mathrm{SO}_n^\circ(q).\langle\phi\rangle$, since $\mathrm{SO}_n^\circ(q) = \mathrm{PSO}_n^\circ(q) = \mathrm{O}_n^\circ(q).\langle\delta\rangle$. Take $g = A\sigma$ for $A \in \mathrm{SO}_n^\circ(q)$ and $\sigma \in \langle\phi\rangle$. Note that ϕ has order e. We will consider the possible orders of σ; note that since δ and ϕ commute we must have that $|\sigma|$ divides 4.

- $|\sigma| = 1$. Then $g = A \in \mathrm{SO}_n^\circ(q) = \mathrm{O}_n^\circ(q)\langle\delta\rangle$.
- $|\sigma| = 2$. Then $\sigma = \phi^{\frac{e}{2}}$ and so $g \in \mathrm{O}_n^\circ(q)\langle\delta\phi^{\frac{e}{2}}\rangle$ or $\mathrm{O}_n^\circ(q)\langle\phi^{\frac{e}{2}}\rangle < \mathrm{O}_n^\circ(q)\langle\phi^{\frac{e}{(4,e)}}\rangle$.
- $|\sigma| = 4$. Then in particular we must have that q is a fourth power, say $q = c^4$. We have that $I_n = g^4 = (A\sigma)^4 = A\sigma^4\sigma^{-3}A\sigma^3\sigma^{-2}A\sigma^2\sigma^{-1}A\sigma = AA^{\sigma^3}A^{\sigma^2}A^{\sigma}$. We have that the spinor norm of A is 1 if and only if $\mu = \prod_{i=1}^{k} v_i f v_i^T$ is a square in $\mathbb{F}_q$, with the v_i as given in Proposition 1.6.27. Similarly the spinor norm of A^{ϕ^i} is 1 if and only if μ^{e^i} is a square in $\mathbb{F}_q$. Then $AA^{\sigma^3}A^{\sigma^2}A^{\sigma} = I_n$ which is an element of $\mathrm{SO}_n^\circ(c)$ with spinor norm 1. Hence μ^r is a square in $\mathbb{F}_c^\times$ where $r = 1 + c + c^2 + c^3 = \frac{c^4-1}{c-1}$. This is the case if and only if μ is a square in $\mathbb{F}_q^\times$, so A has spinor norm 1 and $g \in \mathrm{O}_n^\circ(q)\langle\phi^{\frac{e}{4}}\rangle$.

□

Deleted permutation modules

In the orthogonal case, a large source of examples are the deleted permutation modules.

Definition 3.2.21. A module for a group G is called a *deleted permutation module* if it is the quotient of a permutation module of G by the vector $v = (1, \ldots, 1)$.

The following result allows us to perform computations with deleted permutation modules.

Lemma 3.2.22. *[8, Lemma 4.9.39] Let $G < \mathrm{GL}_n(q)$ consist of the action matrices of a deleted permutation module, with q odd and $n+1 \not\equiv 0 \mod p$. Then G consists of isometries of a symmetric bilinear form. If n is even, then the determinant of the form preserved by G is a square if and only if $n+1$, reduced modulo p, is a square in $\mathbb{F}_q^*$. All elements of G corresponding to even permutations have spinor norm 1. If n is odd and $g \in G$ corresponds to an odd permutation, then the spinor norm of $-g$ is 1 if and only if $\frac{n+1}{2} \mod p$ is a square in $\mathbb{F}_q^*$.*

Definition 3.2.23. A module for a group G is called a *twice deleted permutation module* if it is a constituent of dimension $n-2$ of a deleted permutation module of dimension $n-1$ (which in turn is obtained from a permutation module of dimension n).

The below result applies to a particular example of a twice deleted permutation module, which in particular occurs in dimension 17.

Theorem 3.2.24. *For $n > 6$ odd and $p \mid n$ where p is prime, S_n is a subgroup of $\Omega^\circ_{n-2}(p)$ if $p \equiv 1$ or $3 \mod 8$ and a subgroup of $\mathrm{SO}^\circ_{n-2}(p)$ but not $\Omega^\circ_{n-2}(p)$ if $p \equiv 5$ or $7 \mod 8$.*

Proof. Note that since n is odd, $p \neq 2$. The $(n-2)$-dimensional representation of S_n over the field $\mathbb{F} = \mathbb{F}_p$ for $p \mid n$ is constructed as follows:

Begin with the standard degree n permutation representation of S_n; in other words, for $\sigma \in \mathrm{S}_n$, define the matrix $\rho(\sigma) = (a_{ij})$, where $a_{ij} = \begin{cases} 1 & \text{if } i^\sigma = j, \\ 0 & \text{otherwise.} \end{cases}$

This gives us an n-dimensional module M with basis $e_1, \ldots, e_n$, such that $e_i^\sigma = e_{i^\sigma}$. Note that under this construction, elements corresponding to odd elements of S_n have determinant -1. We multiply these elements by $-I_{n-2}$ to ensure that all matrices have determinant 1, both here and in subsequent computations.

M has a 1-dimensional submodule $K = \langle e_1 + \ldots + e_n \rangle$, giving rise to the deleted permutation module M/K with dimension $n-1$ and basis $e_2 + K, \ldots, e_n + K$. The corresponding matrix for σ is attained by replacing the 1^σ-th row with $(-1, \ldots, -1)$, and then deleting the first row and column of the matrix.

Next, define $f_i := e_i - e_n + K$, and let $N := \langle f_1, \ldots, f_{n-1} \rangle$ be a submodule of M/K. We show that N is irreducible.

Let $0 \neq (f + K) \in N$, so that $f = \sum_{i=1}^{n-1} a_i(e_i - e_n)$, which we can also write as $f = \sum_{i=1}^{n} a_i e_i$ by setting $a_n = -\sum_{i=1}^{n-1} a_i$. Let I be the S_n-module generated by $f + K$. Since $f \notin K$, we cannot have all the coefficients a_i being equal; hence there exist $i \neq j$ such that $a_i \neq a_j$. Then, acting on f by the permutation (i, j) and subtracting f, we get that $(a_j - a_i)e_i + (a_i - a_j)e_j + K \in I$, and since $a_j - a_i \neq 0$ we have that $e_i - e_j + K \in I$. Acting on this element by (j, n), we then have that $e_i - e_n + K \in I$, and then acting by (i, k) for any k with $1 \leq k \leq n-1$ we get that $e_k - e_n + K = f_k \in I$, and so $I = N$ and N is irreducible.

We next show that $f_2, \ldots, f_{n-1}$ are linearly independent; indeed, if

$$0 = \sum_{i=2}^{n-1} a_i(e_i - e_n) + K,$$

we would require that $\sum_{i=2}^{n-1} a_i(e_i - e_n) \in K$; in particular, all the coefficients of the e_i must be equal. The coefficient of e_1 is 0 and the coefficient of e_i is a_i for $2 \leq i \leq n-1$, so that $a_i = 0$ for all i. Thus N is at least $(n-2)$-dimensional. We also have that $\sum_{i=1}^{n-1}(e_i - e_n) = e_1 + e_2 + \ldots + e_{n-1} - (n-1)e_n = e_1 + e_2 + \ldots + e_{n-1} + e_n \in K$ since $p|n$; thus $f_1 \in \langle f_2, \ldots, f_n \rangle$ and N is an irreducible module of dimension $n-2$.

This representation preserves the symmetric bilinear form

$$f = (f_{ij}) = \begin{cases} 1 & \text{if } i = j, \\ \frac{p+1}{2} & \text{if } i \neq j. \end{cases}$$

We know already that the even elements of S_n have spinor norm 1, since $n > 6$ so $[S_n, S_n] = A_n$; hence to determine whether $S_n = A_n.2$ is contained in $\Omega^\circ_{n-2}(p)$, it suffices to find the spinor norm of an odd element of S_n. We choose the element $(1, 2) \in S_n$. The matrix of the $(n-2)$-dimensional representation is given by $g_{ij} = \begin{cases} 1 & \text{if } i = 1, \\ -1 & \text{if } i = j \neq 1, \\ 0 & \text{otherwise.} \end{cases}$

In the context of Lemma 1.6.30 we have

$$a = I_{n-2} - g = \begin{pmatrix} 0 & -1 & -1 & \ldots & -1 \\ 0 & 2 & 0 & \ldots & 0 \\ 0 & 0 & 2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 2 \end{pmatrix}$$

which clearly has rank $n-3$, and a basis of the complement of the nullspace can be taken to be all the standard basis vectors bar the first, yielding the $n-3 \times n-2$

matrix:

$$b = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

Performing the computation $bafb^T$ yields the $n-3 \times n-3$ matrix with 2's on the diagonal and 1's everywhere else. Such a matrix has determinant $n-2$; hence the question reduces to computing the Legendre symbol

$$\left(\frac{n-2}{p}\right) = \left(\frac{-2}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{2}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1, 3 \bmod 8, \\ -1 & \text{if } p \equiv 5, 7 \bmod 8. \end{cases}$$

□

Corollary 3.2.25. *Let $n > 6$ be odd, $\Omega = \Omega^\circ_{n-2}(p)$ with $p \mid n$, let $G = A_n$ and $S = N_\Omega(G)$. Then:*

- *If $p \equiv 1, 3 \bmod 8$, then $S = G.2$ and there are two conjugacy classes of subgroups of Ω isomorphic to S, with trivial class stabiliser.*
- *If $p \equiv 5, 7 \bmod 8$, then $S = G$ and there is a unique conjugacy class of subgroups of Ω isomorphic to S, with class stabiliser $\langle\delta\rangle$.*

In both cases we have a unique $\mathrm{Aut}(\Omega)$*-class of groups G.*

Proof. The result on normalisers is direct from Theorem 3.2.24. Let c denote the number of Ω-classes of G; from Lemma 3.2.10 we have that $c = |C : N_C(G)\Omega|$ where in this case $C = \mathrm{Aut}(\Omega) = \mathrm{Inn}(\Omega).\langle\delta\rangle$. Thus we have

$$\begin{aligned} c = 1 &\iff C = N_C(G)\Omega \iff \delta \in N_C(G) \iff G^\delta = G \\ &\iff \exists g \in \mathrm{SO}^\circ_{n-2}(p) \setminus \Omega^\circ_{n-2}(p) \text{ such that } G^g = G \\ &\iff \exists g \in \mathrm{SO}^\circ_{n-2}(p) \setminus \Omega^\circ_{n-2}(p) \text{ such that } g \in G.2 \\ &\iff G.2 \not\leq \Omega^\circ_{n-2}(p). \end{aligned}$$

The result follows.

□

Other classical groups

For automorphisms of the remaining classical groups (symplectic and orthogonal groups in even dimension), we will typically be able to determine the action of outer automorphisms via more straightforward computations, which we will introduce as we need them.

3.2.4 Containments

Performing the computations up until this point for a given classical group Ω will provide a list of all $\mathcal{S}_1$-candidates, and we now turn our attention to the question of maximality.

Definition 3.2.26. Let Ω be a group, and G, H be groups with representations $\rho_G : G \to \Omega$ and $\rho_H : H \to \Omega$ respectively. Then:

- There is an *abstract containment* of H inside G if there exists a subgroup H' of G such that $H \cong H'$.
- There is a *containment* of H inside G if there is an abstract containment of H inside G with $H \cong H' < G$, and the restriction $\rho_G|_{H'}$ is weakly equivalent to ρ_H.

In other words, a containment of H inside G means that (up to weak equivalence) we have $H\rho_H < G\rho_G < \Omega$; in particular, the representation ρ_H of H does not give rise to a maximal subgroup of Ω.

Definition 3.2.27. Let Ω be a classical group. Then G is $\mathcal{S}_i$*-maximal* if G is an $\mathcal{S}_i$-candidate subgroup of Ω, and there exists no $\mathcal{S}_i$-candidate subgroup K of Ω such that there is a containment of G inside K. We define $\mathcal{S}$*-maximal* in a similar way.

Thus, once we have a list of $\mathcal{S}_1$-candidates, our next aim is to establish which of these groups are $\mathcal{S}_1$-maximal, by determining all possible containments within class $\mathcal{S}_1$. We can rule out the possibility of many containments via Lagrange's theorem, degrees of minimal permutation representations following [35], or direct computations. In the cases where we have an abstract containment $H \cong H' < G$, we make use of the characters corresponding to the representations ρ_G and ρ_H to determine if $\rho_G|_{H'} = \rho_H$. This is usually possible from the character tables in [12] and [32], and by considering elements of certain orders.

3.3 $\mathcal{S}_1$-candidates in dimension 17

3.3.1 Candidates

Theorem 3.3.1. *Let S be an $\mathcal{S}_1$-candidate subgroup of a classical group in dimension 17. Then S^∞ is contained in Table 3.1.*

Proof. Direct from [23] and [24]. □

Table 3.1: $\mathcal{S}_1$-candidates in dimension 17

Group	PmDivs	Out	Ind	# ρ	Stab	Charc	Ch Ring	Theorems
$L_2(17)$	2,3,**17**	2	+	1	2	0	—	3.3.5
$L_2(16)$	**2**,3,5,17	4	+	1	4	$0, 5, 17$	—	3.3.8
$L_2(16)$	**2**,3,5,17	4	+	2	2	$0, 3, 17$	b_5	3.3.9
$L_2(16)$	**2**,3,5,17	4	+	4	1	$0, 17$	b_5, y_{15}	3.3.11
A_{18}	2,3,5,7,11,13,17	2	+	1	2	$0, 5, 7, 11, 13, 17$	—	3.3.4
A_{19}	2,3,5,7,11,13,17,19	2	+	1	2	19	—	3.3.3

Remark 3.3.2. We supply a brief explanation of the content of Table 3.1, along with details of how such information was found:

- 'Group' is the isomorphism class of the group discussed, in Atlas notation. The candidate groups can be found in [23] and [24].
- 'PmDivs' consists of the prime divisors of the order of the group. A number in bold indicates the defining characteristic of the group, if this exists - note that representations over fields in defining characteristic are of type $\mathcal{S}_2$.
- 'Out' is the shape of the outer automorphism group in Atlas notation. In this case these can be determined from [12] or (for the alternating groups) [59, Theorem 2.3].
- 'Ind' denotes the Schur indicator of the representation (recall Definition 3.2.5).
- '$\#\rho$' is the number of weakly equivalent (recall Definition 3.2.2) representations in the character table.
- 'Stab' denotes the stabiliser of the representation under the action of its outer automorphism group.
- 'Charc' is the characteristics in which we have such a representation - this again comes from [23] and [24].

- 'Ch Ring' denotes the character ring of the representation; this is the ring generated by the character values, and we list in this column any irrational generators. These can be easily seen from the character tables, which can usually be found in [12], [60] or GAP. This was not possible for A_{18} and A_{19} in dimension 17; however these characters are easy to find by a result due to Wagner ([56] and [57]), which tells us that the relevant Brauer character χ of A_n (for $n > 8$) over $\mathbb{F}_p$ is given by $\chi(g) = \begin{cases} |\mathrm{Fix}(g)| - 1 & \text{if } p \nmid n, \\ |\mathrm{Fix}(g)| - 2 & \text{if } p \mid n. \end{cases}$

  In particular these characters all have character ring $\mathbb{Z}$.

- 'Theorems' denotes the references for the theorem where the relevant $\mathcal{S}_1$-candidate is considered.

3.3.2 Results

We now determine the $\mathcal{S}_1$-maximal subgroups of $\Omega_{17}^{\circ}(q)$ and its almost simple extensions. Recall from Lemma 1.6.36 that when q is even, $\Omega_{17}^{\circ}(q) \cong \mathrm{S}_{16}(q)$. Hence we only consider the case when q is odd.

From Table 3.1 the candidates are $\mathrm{L}_2(17)$ ($p \neq 2, 3, 17$), $\mathrm{L}_2(16)$ (three times: one with $p \neq 2, 3$, one with $p \neq 2, 5$ and one with $p \neq 2, 3, 5$), A_{18} ($p \neq 2, 3$) and A_{19} ($p = 19$). We will consider these in reverse order.

Proposition 3.3.3. *Let $\Omega = \Omega_{17}^{\circ}(q)$ with $q = p^e$, let $G = A_{19}$ be an $\mathcal{S}_1$-subgroup of Ω, and let $S = N_{\Omega}(G)$. Then $q = p = 19$, $S = G.2$ and there are two Ω-classes of subgroups of Ω isomorphic to S with trivial class stabilizer. The group S is $\mathcal{S}_1$-maximal, there is a single* $\mathrm{Aut}(\Omega)$*-class of groups G, and for no other q is there an $\mathcal{S}_1$-subgroup of $\Omega_{17}^{\circ}(q)$ isomorphic to G.*

Proof. The fact that A_{19} is $\mathcal{S}_1$-maximal follows by Lagrange. The remaining results are direct from Corollary 3.2.25. □

Proposition 3.3.4. *Let $\Omega = \Omega_{17}^{\circ}(q)$ with $q = p^e$, let $G = A_{18}$ be an $\mathcal{S}_1$-subgroup of Ω, and let $S = N_{\Omega}(G)$. Then $q = p \neq 2, 3$, $S = G.2$ and we have two Ω-classes of subgroups isomorphic to S with trivial class stabiliser. If $q = 19$ then S is not $\mathcal{S}_1$-maximal; otherwise, S is $\mathcal{S}_1$-maximal. There is a single* $\mathrm{Aut}(\Omega)$*-class of groups S, and for no other q are there $\mathcal{S}_1$-maximal subgroups of $\Omega_{17}^{\circ}(q)$ isomorphic to A_{18}.*

Proof. The condition on q follows directly from Table 3.1. Since this representation occurs as a deleted permutation module, we can apply Lemma 3.2.22, which says that $G.2 < \Omega_{17}^{\circ}(q)$ for all valid primes p. Hence $N_{\Omega}(G) = G.2$, with trivial class stabiliser. There is a single representation in Theorem 3.3.1; hence by Lemma 3.2.10(i) there is a single conjugacy class of representations in $\mathrm{CGO}^{\circ}_{17}(q)$, and thus a single $\mathrm{Aut}(\Omega)$-class.

The only potential containment involving G is $A_{18} < A_{19}$. This only occurs when $q = 19$; in this case, by Proposition 3.3.3 and from the relevant character tables, we have the containment $A_{18}.2 < A_{19}.2$, so G and $G.2$ are not $\mathcal{S}_1$-maximal in Ω or its extensions by automorphisms. Lagrange gives no other possible containments, so if $q \neq 19$ then $G.2$ is $\mathcal{S}_1$-maximal. □

Proposition 3.3.5. *There are no $\mathcal{S}_1$-maximal subgroups of $\Omega = \Omega^{\circ}_{17}(q)$ or its almost simple extensions involving* $\mathrm{L}_2(17)$ *or its almost simple extensions.*

Proof. The 17-dimensional representation of $\mathrm{L}_2(17)$ appears as a deleted permutation module of the natural action of $\mathrm{L}_2(17)$ on the 18 lines in $\mathbb{F}_{17}^2$. Hence we have $\mathrm{L}_2(17) < A_{18}$. Also, $\mathrm{L}_2(17).2 = \mathrm{PGL}_2(17)$ occurs naturally as a subgroup of $S_{18} = A_{18}.2$ so we have an abstract containment $\mathrm{L}_2(17).2 < A_{18}.2$.

From Lemma 3.2.22, since the 17-dimensional representation of $\mathrm{L}_2(17)$ consists of action matrices of a deleted permutation module, we have that in all cases $G.2 < \Omega$, so that $N_{\Omega}(G) = G.2$ for every valid prime p. From Proposition 3.3.4 we have that $A_{18}.2 < \Omega$, and so it simply remains to establish the restriction of this character, which we denote ρ, to $\mathrm{L}_2(17).2$.

We can obtain the character table of $\mathrm{L}_2(17).2$ from [12, p. 9], and use the corresponding notation. The characters in dimension at most 17 are:

- Two 1-dimensional characters; the trivial character χ_1 and the sign character χ_1'.
- Two 16-dimensional integer-valued characters, which we denote χ_2 and χ_2'.
- Six other 16-dimensional characters, which are not integer-valued.
- Two 17-dimensional characters; namely χ_8 and χ_8'.

The restriction of the character of $A_{18}.2$ to $\mathrm{L}_2(17).2$ must be integer-valued, meaning that it is either one of the 17-dimensional characters, meaning we have a containment, or it is some linear combination of the smaller-dimensional integer-valued characters.

$\mathrm{L}_2(17).2$ contains elements of order 2 which (when considered as elements of $A_{18}.2$) are the product of eight 2-cycles, which therefore have character value 1 and means that the restriction of ρ to $\mathrm{L}_2(17)$ must be either χ_8 or $\chi_1' + \chi_2$. We also have elements in $\mathrm{L}_2(17)$ of order 9 with cycle shape $(9, 9)$, thus with character value -1, which rules out the latter possibility. Hence the restriction is χ_8, which means that we have a containment $\mathrm{L}_2(17).2 < A_{18}.2$. $\square$

Remark 3.3.6. We can do a similar computation with $\mathrm{L}_2(16)$ and A_{18}. In this case, and using the notation of [12, p. 12], the possibilities for the restriction of the 17-dimensional character of A_{18} to $\mathrm{L}_2(16)$ are either $\chi_1 + \chi_{10}$ for χ_1 the trivial character and χ_{10} the integer-valued 16-dimensional character of $\mathrm{L}_2(16)$, or χ_{11} for χ_{11} the integer-valued 17-dimensional character. Considering elements of order 3 gives us that the restriction is $\chi_1 + \chi_{10}$, so we do not have a containment here. A similar argument also rules out the possibility of $\mathrm{L}_2(16) < A_{19}$ in characteristic 19.

Remark 3.3.7. There are three different representations of $\mathrm{L}_2(16)$ to consider. We will denote these by $\mathrm{L}_2(16)_1, \mathrm{L}_2(16)_2$ and $\mathrm{L}_2(16)_3$, based on the order they appear in Table 3.1.

Proposition 3.3.8. *Let $\Omega = \Omega_{17}^{\circ}(q)$ with $q = p^e$, let $G = \mathrm{L}_2(16)_1$ be an $\mathcal{S}_1$-subgroup of Ω, and let $S = N_\Omega(G)$. Then $q = p \neq 2, 3$, and:*

- *(i) If $p \equiv \pm 1 \mod 8$, then $S = G.4$, and we have two Ω-classes of subgroups with trivial class stabiliser.*
- *(ii) If $p \equiv \pm 3 \mod 8$, then $S = G.2$, and we have a single Ω-class of subgroups with class stabiliser $\langle \delta \rangle$.*

The group S is $\mathcal{S}_1$-maximal, there is a single $\mathrm{Aut}(\Omega)$-class of groups S, and the only other situations where we have $\mathcal{S}_1$-subgroups of $\Omega_{17}^{\circ}(q)$ isomorphic to G are described in Proposition 3.3.9 and Proposition 3.3.11.

Proof. Since we can see from Table 3.1 that the character ring contains no irrationalities, we always have $G < \Omega_{17}^{\circ}(p)$ for valid primes p (i.e. $p \neq 2, 3$). Calculations contained in the computer file `l216d171calc` perform the spinor norm calculation in characteristic 0 for an element which induces the 4 automorphism on G. The calculations include checks that the group constructed is $G.4$ and that it preserves the given form.

We then construct matrices following Lemma 1.6.30 to compute the spinor norm. The calculations in `l216d171calc` show that the spinor norm is 1 if and only if 2 is a square modulo p, which occurs precisely when $p \equiv \pm 1 \mod 8$. If the element g inducing the 4 automorphism of G has spinor norm -1, then g^2 inducing the 2 automorphism of G must have spinor norm 1.

There is a single representation in Theorem 3.3.1, and hence a single $\mathrm{Aut}(\Omega)$-class by Lemma 3.2.10. Standard computations confirm the number of Ω-classes.

When there are two classes, δ interchanges them, and so the results on the class stabiliser follow as well. $\square$

Proposition 3.3.9. *Let $\Omega = \Omega_{17}^{\circ}(q)$ with $q = p^e$, let $G = \mathrm{L}_2(16)_2$ be an $\mathcal{S}_1$-subgroup of Ω, and let $S = N_{\Omega}(G)$. Then provided $p \neq 2, 5$, we have:*

(i) If $p \equiv \pm 2 \mod 5$ then $q = p^2$, $S = G.2$ and there are two Ω-classes of subgroups isomorphic to S. There is a single $\mathrm{Aut}(\Omega)$-class. We do not currently have a proof of what the class stabilisers are; see Remark 3.3.10.

(ii) If $p \equiv \pm 1 \mod 5$ and $p \equiv \pm 1 \mod 8$, then $q = p$, $S = G.2$ and there are two Ω-classes of subgroups isomorphic to S with trivial class stabilisers. There are two $\mathrm{Aut}(\Omega)$-classes.

(iii) If $p \equiv \pm 1 \mod 5$ and $p \equiv \pm 3 \mod 8$, then $q = p$, $S = G$, and there is a single Ω-class of subgroups isomorphic to S, with class stabiliser $\langle \delta \rangle$. There are two $\mathrm{Aut}(\Omega)$-classes.

The group S is $\mathcal{S}_1$-maximal and the only other situations where we have subgroups of $\Omega_{17}^{\circ}(q)$ isomorphic to G are described in Proposition 3.3.8 and Proposition 3.3.11.

Proof. The order of the field of definition depends on the existence of the quadratic irrationality b_5, which lies in $\mathbb{F}_p$ if $p \equiv 1, 4 \mod 5$, and $\mathbb{F}_{p^2}$ but not $\mathbb{F}_p$ if $p \equiv 2, 3 \mod 5$.

We perform similar calculations to Proposition 3.3.8 in `l216d172calc`, although this time the characteristic 0 group we are dealing with involves the quadratic irrationality b_5.

The computation tells us that the matrix relating to the 2 automorphism of G has spinor norm 1 if and only if 10 is a square in the field.

If $q = p^2$ then 10 is contained in the prime field and so is always a square; hence here we can always realise the 2 automorphism over Ω. If $q = p$, then in particular $p \equiv \pm 1 \mod 5$; thus, from the Legendre symbol for 5 we can see that 5 is always a square, and the Legendre symbol for 2 gives the congruences as described above.

It follows from the character table in [12] that there are two orbits of $\mathrm{Out}(G)$ on the weak equivalence class of representations, and hence by Lemma 3.2.10 we have two conjugacy classes in $\mathrm{CGO}_{17}^{\circ}(q)$. If $q = p$ then $\mathrm{PGCO}_{17}^{\circ}(q) = \mathrm{Aut}(\Omega)$ and hence we have two $\mathrm{Aut}(\Omega)$ classes. If $q = p^2$ then the nontrivial field automorphism of Ω interchanges the two conjugacy classes of representations so there is a single $\mathrm{Aut}(\Omega)$-class.

For the class stabiliser calculations, note that if $G.2 < \Omega$, then $N_C(G) < Z\Omega$, where Z is the group of scalar matrices. Then $N_C(G) = N_C(G)\Omega$ and so, by Lemma 3.2.10 the number of Ω-classes is $|C : N_C(G)\Omega| = |C : \Omega| = 2$. Likewise, if $G.2 \not< \Omega$, then $N_C(G)\Omega = C$ and so we have a single Ω-class in this case.

In case (ii) we have two classes permuted by δ and no further outer automorphisms exist, so the class stabiliser is trivial.

In case (iii) we have one class; hence the class stabiliser is $\langle\delta\rangle$, the whole outer automorphism group of Ω. $\square$

Remark 3.3.10. Computer calculations give us the conjecture that in Proposition 3.3.9(i), the class stabiliser should be $\langle\phi\rangle$ when $p \equiv \pm 1 \mod 8$ and $\langle\phi\delta\rangle$ when $p \equiv \pm 3 \mod 8$.

Proposition 3.3.11. *Let $\Omega = \Omega^\circ_{17}(q)$ with $q = p^e$, let $G = \mathrm{L}_2(16)_3$ be an $\mathcal{S}_1$-subgroup of Ω, and let $S = N_\Omega(G)$. Then provided $p \neq 2, 3, 5$, we have $S = G$, there are two Ω-classes of subgroups isomorphic to G, and:*

(i) If $p \equiv \pm 1 \mod 15$ then $q = p$, with trivial class stabiliser.

(ii) If $p \equiv \pm 4 \mod 15$ then $q = p^2$, with class stabiliser $\langle\phi\rangle$.

(iii) If $p \equiv \pm 2, \pm 7 \mod 15$ then $q = p^4$, with class stabiliser $\langle\phi\rangle$.

The group S is $\mathcal{S}_1$-maximal, there is a single $\mathrm{Aut}(\Omega)$-class of groups S, and the only other situations where we have subgroups of $\Omega^\circ_{17}(q)$ isomorphic to G are described in Proposition 3.3.8 and Proposition 3.3.9.

Proof. The representation $\mathrm{L}_2(16)_3$ involves the irrationalities b_5 and $y_{15} = \theta + \theta^{-1}$, where θ is a primitive 15-th root of unity. Recall from Section 1.9.2 that y_{15} exists over $\mathbb{F}_q$ for $q = p^e$ for e as small as possible, where

$$e = \begin{cases} 1 & \text{if } p \equiv 1, 14 \bmod 15, \\ 2 & \text{if } p \equiv 4, 11 \bmod 15, \\ 4 & \text{if } p \equiv 2, 7, 8, 13 \bmod 15. \end{cases}$$

It turns out that, whenever y_{15} exists in a field, so does b_5; hence the above is sufficient to determine the minimal field of representation of $\mathrm{L}_2(16)_3$.

In all cases, we have two Ω-classes, and δ interchanges these two classes. From the character table in [12] we see that the four weakly equivalent representations are permuted cyclically by the field automorphism of G; hence by Lemma 3.2.10 we have a single $\mathrm{Aut}(\Omega)$-class.

For (i), since $\mathrm{Out}(\Omega) = \langle \delta \rangle$, the above is enough to confirm that the class stabiliser is trivial.

For (ii), since $G.2 \setminus G$ contains involutions, we can apply Lemma 3.2.19 to conclude that the class stabiliser here is $\langle \phi \rangle$.

For (iii), let α be an element of the class stabiliser. We have 2 Ω-classes and 8 outer automorphisms of Ω, so the stabiliser must have 4 elements; so the stabiliser is either C_4 or $C_2 \times C_2$. If it were $C_2 \times C_2$, then we would have $|\alpha| = 2$ for every choice of α in the stabiliser. Then by Lemma 3.2.19 we would have that $\alpha \in \langle \delta \rangle$ or $\alpha \in \langle \phi \rangle$. This would force the stabiliser to be $\langle \delta, \phi^2 \rangle$, which is not possible as δ does not stabilise either class. Hence we have that the stabiliser has shape C_4; this leaves us with the option of either $\langle \phi \rangle$ or $\langle \phi\delta \rangle$. Since $G.4 \setminus G$ contains elements of order 4, we can apply Lemma 3.2.20 which tells us that the stabiliser cannot contain $\phi\delta$, so we must have the class stabiliser is $\langle \phi \rangle$ as claimed. $\square$

3.3.3 Summary

We summarise the results of the previous section.

Remark 3.3.12. We use [8, Convention 4.10.1] for recording the results in this section and in subsequent summaries. For ease of reference we reproduce this convention here:

(i) The group Ω is a quasisimple classical group, $Z = Z(\Omega)$, $\overline{\Omega} = \Omega/Z$ and G is a group with $\overline{\Omega} \leq G \leq \mathrm{Aut}(\overline{\Omega})$.

(ii) The structure of a proper subgroup S of Ω with $Z < S$ is specified, and $\overline{S} := S/Z$. This subgroup represents a single conjugacy class of subgroups of Ω under the action of $\mathrm{Aut}(\overline{\Omega})$.

(iii) The values of $q = p^e$ for which this list item may represent $\mathcal{S}_i$-maximal subgroups of G are specified. Different values of q may correspond to different cases for Ω.

(iv) The stabiliser of the conjugacy class of S in Ω under the action of $\mathrm{Out}(\overline{\Omega})$ is specified as a subgroup of $\mathrm{Out}(\overline{\Omega})$, with automorphisms as given in Section 1.6.

(v) For the specified values of q, the list item represents $\mathcal{S}_i$-maximal subgroups of G only if $G/\overline{\Omega}$ is a subgroup of the class stabiliser. The default assumption is that this is the case if and only if $G/\overline{\Omega}$ is a subgroup of the class stabiliser.

In cases where this is not true (that is, for the novel maximal subgroups), the subgroups of the class stabiliser for which it is true are specified.

(vi) If the list item does represent $\mathcal{S}_i$-maximal subgroups of G, then one such subgroup is $N_G(\overline{S})$. Representatives of the G-classes of subgroups represented by this item are obtained by conjugating $N_G(\overline{S})$ by coset representatives of $N_T(G)$ in $N_{\mathrm{Aut}(\overline{\Omega})}(G)$, where T is the inverse image in $\mathrm{Aut}(\overline{\Omega})$ of the class stabiliser.

Theorem 3.3.13. *Let G and Ω be as in Remark 3.3.12 with $\Omega = \Omega^\circ_{17}(q)$. Then representatives of the conjugacy classes of $\mathcal{S}_1$-maximal subgroups of G are described in the list below:*

Proof. See Section 3.3.2. □

(i) $S = A_{19}.2 < \Omega^\circ_{17}(19)$ with trivial class stabiliser.

(ii) $S = A_{18}.2 < \Omega^\circ_{17}(p)$ with $p \neq 2, 3, 19$, with trivial class stabiliser.

(iii) $S = \mathrm{L}_2(16).4$ with $p \equiv \pm 1 \mod 8$, or $S = \mathrm{L}_2(16).2$ with $p \neq 3$ and $p \equiv \pm 3 \mod 8$, with $S < \Omega^\circ_{17}(p)$ in both cases. The class stabiliser is trivial when $S = \mathrm{L}_2(16).4$ and $\langle \delta \rangle$ when $S = \mathrm{L}_2(16).2$.

(iv) $S = \mathrm{L}_2(16)$ if $p \equiv \pm 1 \mod 5$ and $p \equiv \pm 3 \mod 8$, or $S = \mathrm{L}_2(16).2$ otherwise. If $p \equiv \pm 1 \mod 5$ then $q = p$, and otherwise $q = p^2$. When $q = p$ the class stabiliser is trivial when $p \equiv \pm 1 \mod 8$ and $\langle \delta \rangle$ when $p \equiv \pm 3 \mod 8$. We conjecture that when $q = p^2$, the class stabiliser is $\langle \phi \rangle$ when $p \equiv \pm 1 \mod 8$ and $\langle \phi\delta \rangle$ when $p \equiv \pm 3 \mod 8$.

(v) $S = \mathrm{L}_2(16)$ with $p \neq 2, 3, 5$. We have $S < \Omega^\circ_{17}(p)$ if $p \equiv \pm 1 \mod 15$, $S < \Omega^\circ_{17}(p^2)$ if $p \equiv \pm 4 \mod 15$, and $S < \Omega^\circ_{17}(p^4)$ if $p \equiv \pm 2, \pm 7 \mod 15$. The class stabiliser is trivial if $p \equiv \pm 1 \mod 15$ and $\langle \phi \rangle$ otherwise.

3.4 $\mathcal{S}_1$-candidates in dimension 16

3.4.1 Candidates

Table 3.2 contains the candidates in dimension 16; these were found using the same method used to construct Table 3.1. The table is ordered first by Schur indicator and then by the order of the group from smallest to largest. See Remark 3.3.2 for more detail about the information contained in each column.

Theorem 3.4.1. *Let S be an $\mathcal{S}_1$-candidate subgroup of a classical group in dimension 16. Then S^∞ is contained in Table 3.2.*

Proof. Direct from [23] and [24]. □

Table 3.2: $\mathcal{S}_1$-candidates in dimension 16

Group	PmDivs	Out	Ind	# ρ	Stab	Charc	Ch Ring	Theorems
$2^{\cdot}\mathrm{L}_2(17)$	2,3,**17**	2	-	1	2	0	—	3.4.5
$2^{\cdot}\mathrm{L}_2(17)$	2,3,**17**	2	-	3	2	0	y_9	3.4.6
$2^{\cdot}A_7$	2,3,5,7	2	-	1	2	7	—	3.4.3
$2^{\cdot}A_8$	2,3,5,7	2	-	1	2	7	—	3.4.3
A_{18}	2,3,5,7,11,13,17	2	-	1	2	2	—	3.4.2
$\mathrm{L}_2(17)$	2,3,**17**	2	+	1	2	$0, 2, 3$	—	3.4.18
$\mathrm{L}_2(17)$	2,3,**17**	2	+	3	2	$0, 2$	y_9	3.4.19
$\mathrm{L}_2(16)$	**2**,3,5,17	4	+	1	4	$0, 3, 5$	—	3.4.17
$\mathrm{L}_3(3)$	2,**3**,13	2	+	1	2	13	—	3.4.16
M_{11}	2,3,5,11	1	+	1	1	11	—	3.4.15
$2^{\cdot}\mathrm{Sz}(8)$	**2**,5,7,13	1	+	1	1	13	y_7	3.4.14
M_{12}	2,3,5,11	2	+	1	2	11	—	3.4.13
A_{10}	2,3,5,7	2	+	1	2	2	—	3.4.12
$2^{\cdot}A_{10}$	2,3,5,7	2	+	1	2	$0, 3, 7$	—	3.4.11
$2^{\cdot}A_{11}$	2,3,5,7,11	2	+	1	2	11	—	3.4.10
A_{17}	2,3,5,7,11,13,17	2	+	1	2	$0, 2, 3, 5, 7, 11, 13$	—	3.4.9
A_{18}	2,3,5,7,11,13,17	2	+	1	2	3	—	3.4.8
$\mathrm{L}_3(3)$	2,**3**,13	2	∘	4	1	$0, 2$	d_{13}	3.4.27
M_{11}	2,3,5,11	1	∘	2	1	$0, 2, 5$	b_{11}	3.4.24
$2^{\cdot}\mathrm{L}_2(31)$	2,3,5,**31**	2	∘	2	1	$0, 3, 5$	b_{31}	3.4.26
$4_2^{\cdot}\mathrm{L}_3(4)$	**2**,3,5,7	2^2	∘	2	2_2	3	i, b_7	3.4.25
M_{12}	2,3,5,11	2	∘	2	1	$0, 2, 5$	b_{11}	3.4.24
$4^{\cdot}M_{22}$	2,3,5,7,11	2	∘	2	1	7	i, b_{11}	3.4.23
A_{11}	2,3,5,7,11	2	∘	2	1	2	b_{11}	3.4.21
$2^{\cdot}A_{11}$	2,3,5,7,11	2	∘	2	1	$0, 3, 5, 7$	b_{11}	3.4.22
A_{12}	2,3,5,7,11	2	∘	2	1	2	z_3, b_{11}, b_{35}	3.4.21
$2^{\cdot}A_{12}$	2,3,5,7,11	2	∘	2	1	3	i_2, i_5, b_{11}, b_{35}	3.4.20

3.4.2 Results

Symplectic case

We now determine the $\mathcal{S}_1$-maximal subgroups of $\mathrm{Sp}_{16}(q)$ and its almost simple extensions. From Table 3.2 the candidates are $2^{\cdot}\mathrm{L}_2(17)$ (twice, both with $p \neq 2, 3, 17$), $2^{\cdot}A_7$ ($p = 7$), $2^{\cdot}A_8$ ($p = 7$) and A_{18} ($p = 2$). We will denote the two different weak equivalence classes of $2^{\cdot}\mathrm{L}_2(17)$ by $2^{\cdot}\mathrm{L}_2(17)_1$ and $2^{\cdot}\mathrm{L}_2(17)_2$ respectively, depending on the order they appear in the table. We will consider the candidates in reverse order in this section.

Recall the outer automorphism group of $\mathrm{Sp}_{16}(p^e)$ from Section 1.6.6.

Proposition 3.4.2. *Let $\Omega = \mathrm{Sp}_{16}(q)$ with $q = p^e$, let $G = A_{18}$, and let $S = N_\Omega(G)$. Then $q = p = 2$, $S = G.2$ and we have a single Ω-class of subgroups isomorphic to*

S, with trivial class stabiliser. The group S is $\mathcal{S}_1$-maximal, there is a single $\mathrm{Aut}(\Omega)$*-class of groups S, and for no other q are there $\mathcal{S}_1$-subgroups of Ω isomorphic to G.*

Proof. Table 3.2 shows that we only have a 16-dimensional character in characteristic 2, and since the character ring is $\mathbb{Z}$ we have that $q = p$ here. A short computer calculation in `a18d16f2calc` shows that $G.2$ preserves a symplectic form, and hence $G.2 < \mathrm{Sp}_{16}(2)$. Since we have a single representation, and since the outer automorphism group of $\mathrm{Sp}_{16}(2)$ is trivial, the rest of the results follow. Lagrange rules out all possible containments. $\square$

Proposition 3.4.3. *Let $\Omega = \mathrm{Sp}_{16}(q)$, and $q = p^e$.*

(i) The groups $2^{\cdot}A_7$ and $2^{\cdot}A_7.2$ are not $\mathcal{S}_1$-maximal in Ω or its extensions by automorphisms for any q.

(ii) Let $G = 2^{\cdot}A_8$ be an $\mathcal{S}_1$-maximal subgroup of Ω, and let $S = N_\Omega(G)$. Then $q = p = 7$, $S = G$ and there is a single Ω-class of subgroups isomorphic to S, with class stabiliser $\langle\delta\rangle$. The group S is $\mathcal{S}_1$-maximal, there is a single $\mathrm{Aut}(\Omega)$*-class of groups S, and for no other q are there $\mathcal{S}_1$-subgroups of Ω isomorphic to G.*

Proof. Both $2^{\cdot}A_7$ and $2^{\cdot}A_8$ have a single 16-dimensional representation, defined only in characteristic 7 with trivial character ring, and the extensions $2^{\cdot}A_7.2$ and $2^{\cdot}A_8.2$ both require the irrationalities r_5 and r_6, neither of which exist in $\mathbb{F}_7$. Hence we have that both are only $\mathcal{S}_1$-candidate subgroups when $q = p = 7$, with both $2^{\cdot}A_7$ and $2^{\cdot}A_8$ being self-normalising in $\mathrm{Sp}_{16}(q)$. For $G = 2^{\cdot}A_8$, this tells us that $N_\Omega(G) = G$, meaning we have a unique Ω-class of subgroups of Ω isomorphic to G, with class stabiliser $\langle\delta\rangle$.

We clearly have an abstract containment $2^{\cdot}A_7 < 2^{\cdot}A_8$, and the character values found in [32] show that the restriction of the representation of $2^{\cdot}A_8$ to $2^{\cdot}A_7$ is the irreducible 16-dimensional representation of $2^{\cdot}A_7$, so we have a containment of $\mathcal{S}_1$ subgroups. Further it follows from the previous paragraph that δ induces the 2 automorphism on both $2^{\cdot}A_7$ and $2^{\cdot}A_8$, and the 2 automorphism of $2^{\cdot}A_8$ induces the 2 automorphism of $2^{\cdot}A_7$, so there are no $\mathcal{S}_1$-maximal subgroups of extensions of Ω isomorphic to $2^{\cdot}A_7.2$. $\square$

Proposition 3.4.4. *There is no abstract containment $2^{\cdot}\mathrm{L}_2(17) < A_{18}$.*

Proof. By looking at the maximal subgroups of $2^{\cdot}\mathrm{L}_2(17)$ we see that the largest subgroup is of index 18. Thus this subgroup must contain the central element of order 2 in $2^{\cdot}\mathrm{L}_2(17)$, giving us a permutation representation on 18 points of $\mathrm{L}_2(17)$ but not $2^{\cdot}\mathrm{L}_2(17)$. Hence the result follows. □

Proposition 3.4.5. *Let* $\Omega = \mathrm{Sp}_{16}(q)$ *with* $q = p^e$*, let* $G = 2^{\cdot}\mathrm{L}_2(17)_1$*, and let* $S = N_\Omega(G)$*. Then:*

(i) If $p \equiv \pm 1 \mod 12$ *then* $q = p$*,* $S = G.2$ *and there are two* Ω*-classes of subgroups isomorphic to* S*, with trivial class stabiliser.*

(ii) If $p \equiv \pm 5 \mod 12$ *and* $p \neq 17$ *then* $q = p$*,* $S = G$ *and there is a single* Ω*-class of subgroups isomorphic to* S*, with class stabiliser* $\langle\delta\rangle$*.*

In both cases the group S *is* $\mathcal{S}_1$*-maximal, there is a unique* $\mathrm{Aut}(\Omega)$*-class of groups* S*, and the only other situation where we may have an* $\mathcal{S}_1$*-subgroup of* $\mathrm{Sp}_{16}(q)$ *isomorphic to* G *is described in Proposition 3.4.6.*

Proof. Table 3.2 gives us that $q = p \neq 2, 3, 17$. From [12] we see that there is a single such representation, and for this representation that elements of $G.2 \setminus G$ are isometries, but their character values involve r_3, which lies in $\mathbb{F}_p$ if and only if $p \equiv \pm 1 \mod 12$. Multiplying elements of $G.2 \setminus G$ by a suitable scalar would give us the isoclinic group $G.2^-$, but this does not consist of isometries of the form, since the only scalars which preserve the symplectic form are $\pm I_{16}$.

Then standard results give the number of Ω classes and the class stabilisers. The only possible containment involving G is considered in Proposition 3.4.4. □

Proposition 3.4.6. *Let* $\Omega = \mathrm{Sp}_{16}(q)$ *with* $q = p^e$*, let* $G = 2^{\cdot}\mathrm{L}_2(17)_2$*, and let* $S = N_\Omega(G)$*. Then provided* $p \neq 2, 3, 17$ *we have:*

(i) If $p \equiv \pm 1 \mod 36$ *then* $q = p$ *and* $S = G.2$*. There are three* $\mathrm{Aut}(\Omega)$*-classes of subgroups isomorphic to* S*, each splitting into two* Ω*-classes of subgroups with trivial class stabiliser.*

(ii) If $p \equiv \pm 11, \pm 13 \mod 36$ *then* $q = p^3$ *and* $S = G.2$*. There is a single* $\mathrm{Aut}(\Omega)$*-class of subgroups isomorphic to* S*, and six* Ω*-classes of subgroups, with trivial class stabiliser.*

(iii) If $p \equiv \pm 17 \mod 36$ *then* $q = p$ *and* $S = G$*. There are three* $\mathrm{Aut}(\Omega)$*-classes of subgroups isomorphic to* S*, each with a single* Ω*-class of subgroups with class stabiliser* $\langle\delta\rangle$*.*

(iv) If $p \equiv \pm 5, \pm 7 \mod 36$ *then* $q = p^3$ *and* $S = G$*. There is a single* $\mathrm{Aut}(\Omega)$*-class of subgroups isomorphic to* S*, and three* Ω*-classes of subgroups with class stabiliser* $\langle\delta\rangle$*.*

The group S is $\mathcal{S}_1$-maximal and the only other situation where we may have an $\mathcal{S}_1$-subgroup of $\mathrm{Sp}_{16}(q)$ *isomorphic to G is described in Proposition 3.4.5.*

Proof. From Table 3.2 we have that the character ring involves the irrationality y_9, which is defined over $\mathbb{F}_p$ if $p \equiv 1, 3, 8 \mod 9$ and $\mathbb{F}_{p^3}$ otherwise. From [12] we can see that the representation of $G.2$ consists of isometries (since it has Schur indicator $-$) but the character values on elements of $G.2 \backslash G$ involve y_{36}, which, given that $p \neq 2, 3$, exists over $\mathbb{F}_q$ if and only if $q \equiv \pm 1 \mod 36$; hence we obtain the given conditions on p. The isoclinic variant $G.2^-$ obtained by multiplying elements of $G.2 \setminus G$ by a scalar of order 4 (if such a scalar exists) does not preserve the symplectic form, since the only scalars to preserve the form are $\pm I_{16}$.

The three representations are fixed by δ and permuted by ϕ when ϕ is non-trivial. If the 2 automorphism of G is realisable over Ω, then each representation splits into two conjugacy classes which are induced by δ; otherwise, δ stabilises each of these classes and induces the 2 automorphism of G. Thus the results on the number of classes and the class stabilisers follow. The only possible containment involving S is considered in Proposition 3.4.4. □

Orthogonal case

We now determine the $\mathcal{S}_1$-maximal subgroups of $\Omega^{\pm}_{16}(q)$ and its almost simple extensions. From Table 3.2 the candidates are $\mathrm{L}_2(17)$ (twice; one with $p \neq 17$ and one with $p \neq 3, 17$), $\mathrm{L}_2(16)$ ($p \neq 2, 17$), $\mathrm{L}_3(3)$ ($p = 13$), M_{11} ($p = 11$), $2^{\cdot}\mathrm{Sz}(8)$ ($p = 13$), M_{12} ($p = 11$), A_{10} ($p = 2$), $2^{\cdot}A_{10}$ ($p \neq 2, 5$), $2^{\cdot}A_{11}$ ($p = 11$), A_{17} ($p \neq 17$) and A_{18} ($p = 3$). We denote the two copies of $\mathrm{L}_2(17)$ by $\mathrm{L}_2(17)_1$ and $\mathrm{L}_2(17)_2$ depending on the order they appear in Table 3.2. We will consider these groups in reverse order.

Remark 3.4.7. Recall the presentations of $\mathrm{Out}(\mathrm{O}^+_{16}(q))$ and $\mathrm{Out}(\mathrm{O}^-_{16}(q))$ from Section 1.6.7. In particular, when q is odd we have that:

(i) The only conjugacy class that lies in $\mathrm{PSO}^+_{16}(q) \setminus \mathrm{O}^+_{16}(q)$ has representative δ'.

(ii) The only conjugacy class that lies in $\mathrm{PGO}^{\pm}_{16}(q) \setminus \mathrm{PSO}^{\pm}_{16}(q)$ has representative γ.

(iii) The only conjugacy classes that lie in $\mathrm{PCGO}^{\pm}_{16}(q) \setminus \mathrm{PGO}^{\pm}_{16}(q)$ have representatives δ and $\gamma\delta$.

(iv) When in the orthogonal plus-type case, note that $\delta^2 = 1$, whereas $(\gamma\delta)^2 = \delta'$. Thus, if a matrix g lies in $\mathrm{CGO}^+_{16}(q) \setminus \mathrm{GO}^+_{16}(q)$ and induces an outer automorphism of order 2 on an $\mathcal{S}_1$-candidate, then g must induce the δ automorphism;

otherwise, δ' would induce an inner automorphism on G. Since the representation of G is absolutely irreducible, this would mean that g^2 was a scalar multiple of an element of Ω; however all scalars in $\mathrm{GO}_{16}^{+}(q)$ have spinor norm 1, so this is impossible.

(v) In the orthogonal minus-type case, both δ and $\delta\gamma$ have order 2, so the above argument does not work.

Thus if a matrix inducing an automorphism of G lies inside $\mathrm{PGO}_{16}^{\pm}(q)$, then we know the class stabiliser without further computation.

When q is even, we will typically perform the computations directly in Magma.

Proposition 3.4.8. *Let $\Omega = \Omega_{16}^{\epsilon}(q)$ with $q = p^e$, let $G = A_{18}$ be an $\mathcal{S}_1$-subgroup of Ω and let $S = N_\Omega(G)$. Then $q = p = 3$, $\epsilon = -$, $S = G$ and we have two Ω-classes of subgroups isomorphic to S, with class stabiliser $\langle\gamma\rangle$. The group S is $\mathcal{S}_1$-maximal, there is a single* $\mathrm{Aut}(\Omega)$*-class of groups S and for no other q is there an $\mathcal{S}_1$-subgroup of $\Omega_{16}^{\pm}(q)$ isomorphic to A_{18}.*

Proof. From Table 3.2 we see that this representation in dimension 16 is only defined in characteristic 3. The character ring is $\mathbb{Z}$ and a brief Magma calculation found in `a18d16calc` shows that $G < \Omega_{16}^{-}(3)$. Recalling that $G.2 = \mathrm{Sym}(18)$ and that G is a twice deleted permutation representation, we have that the conjugacy class of $(1,2)$ lies inside $G.2 \setminus G$, consists of elements of order 2 and has character value $|\mathrm{Fix}((1,2))| - 2 = 14$; this must therefore have eigenvalues consisting of 15 1's and one -1, and so this has determinant -1. Thus $G.2 \not< \mathrm{SO}_{16}^{-}(3)$. From `a18d16calc` we see that $G.2$ preserves the form, so $G.2 \setminus G \subset \mathrm{GO}_{16}^{-}(3) \setminus \mathrm{SO}_{16}^{-}(3)$. Hence the class stabiliser is $\langle\gamma\rangle$ from Remark 3.4.7. Thus there are two classes of subgroups of Ω isomorphic to G.

There are no possible containments involving A_{18}, so A_{18} is an $\mathcal{S}_1$-maximal subgroup of Ω. □

Proposition 3.4.9. *Let $\Omega = \Omega_{16}^{\epsilon}(q)$ with $q = p^e$, let $G = A_{17}$, and let $S = N_\Omega(G)$. Then $q = p \neq 17$, $S = G$ and we have:*

(i) If $p \equiv \pm 1, \pm 2, \pm 4, \pm 8 \mod 17$ then $\epsilon = +$, and there are $(p-1,2)^2$ Ω-classes of subgroups isomorphic to S, with class stabiliser $\langle\gamma\rangle$.

(ii) If $p \equiv \pm 3, \pm 5, \pm 6, \pm 7 \mod 17$ then $\epsilon = -$ and there are two Ω-classes of subgroups isomorphic to S, with class stabiliser $\langle\gamma\rangle$.

If $p = 3$ then G and $G.2$ are not $\mathcal{S}_1$-maximal in Ω or any extension of Ω by automorphisms. If $p \neq 3$ then the group S is $\mathcal{S}_1$-maximal, there is a single $\mathrm{Aut}(\Omega)$*-class of groups S and for no other q is there an $\mathcal{S}_1$-subgroup of $\Omega^{\pm}_{16}(q)$ isomorphic to G.*

Proof. The 16-dimensional module for A_{17} is a deleted permutation module; hence when p is odd we can apply Lemma 3.2.22 to see that G preserves an orthogonal plus-type form if p is a square modulo 17, and an orthogonal minus-type form otherwise. It follows from the proof of Lemma 3.2.22 in [8, Lemma 4.9.39] that the images of odd elements of S_{17} preserve the form and have determinant -1; hence by Remark 3.4.7 the class stabiliser is $\langle\gamma\rangle$. Separate calculations for $p = 2$ in `a17d16p2` confirm the type of the form preserved by G and the class stabiliser.

There is an abstract containment $A_{17} < A_{18}$, with the former a point stabiliser of the latter. From the previous paragraph and Proposition 3.4.8 we see that characteristic 3 is the only characteristic where both these groups exist in dimension 16, and both preserve an orthogonal minus form. It is clear from the characters that the restriction of the character of A_{18} to the point stabiliser A_{17} is the irreducible 16-dimensional character of A_{17}, and similarly γ stabilises both A_{18} and A_{17}, with the 2 automorphism of A_{18} inducing the 2 automorphism of A_{17}. Hence A_{17} and $A_{17}.2$ are not maximal in any extension of Ω by automorphisms. There are no other possible containments. $\square$

Proposition 3.4.10. *Let $\Omega = \Omega^{\epsilon}_{16}(q)$ with $q = p^e$, let $G = 2^{\cdot}A_{11}$ be an $\mathcal{S}_1$-subgroup of Ω, and let $S = N_{\Omega}(G)$. Then $q = p = 11$, $\epsilon = +$, $S = G.2$, and we have eight Ω-classes of subgroups isomorphic to S, with trivial class stabiliser. The group S is $\mathcal{S}_1$-maximal, there is a single* $\mathrm{Aut}(\Omega)$*-class of groups S, and for no other q are there $\mathcal{S}_1$-subgroups of $\Omega^{\pm}_{16}(q)$ isomorphic to G.*

Proof. The condition on q is direct from Table 3.2; the rest of the claims, excluding maximality, follow from a computer calculation in `2a11d16f11calc`.

For maximality, the only potential abstract containment is $2^{\cdot}A_{11} < A_{17}$. If $2^{\cdot}A_{11}$ could be expressed as a subgroup of A_{17} then we would have a corresponding 17-dimensional representation of $2^{\cdot}A_{11}$ given by the number of fixed points for a representative of each conjugacy class. However the only irreducible representations of $2^{\cdot}A_{11}$ of degree at most 17 are the trivial representation and the 16-dimensional representation. Thus the only possibility for the restriction of the 17-dimensional character of A_{17} to $2^{\cdot}A_{11}$ is the direct sum of the 16-dimensional character and the trivial character. However the character values differ, for instance on elements of order 3, so we do not have an abstract containment here. Hence S is $\mathcal{S}_1$-maximal. $\square$

Proposition 3.4.11. *Let $\Omega = \Omega^{\epsilon}_{16}(q)$ with $q = p^e$, let $G = 2^{\cdot}A_{10}$, and let $S = N_{\Omega}(G)$. Then $q = p \neq 2, 5$, and $\epsilon = +$. We have:*

(i) If $p \equiv \pm 1 \mod 10$ then $S = G.2$, and we have eight Ω-classes isomorphic to S with trivial class stabiliser.

(ii) If $p \equiv \pm 3 \mod 10$ then $S = G$, and we have four Ω-classes isomorphic to S with class stabiliser $\langle \delta \rangle$.

If $p = 11$ then S is not $\mathcal{S}_1$-maximal in Ω or any extension of Ω by automorphisms, whilst if $p \neq 11$ then S is $\mathcal{S}_1$-maximal. There is a single $\mathrm{Aut}(\Omega)$-class of subgroups isomorphic to S, and for no other q is there an $\mathcal{S}_1$-subgroup of $\Omega^{\pm}_{16}(q)$ isomorphic to S.

Proof. A computer calculation in `2a10d16calc` shows that the orthogonal form preserved is always of plus type. From the character tables in [12] and [32] we see that the 16-dimensional irreducible character of $G.2 \setminus G$ takes the value 0 on all conjugacy classes except one, which requires the irrationality r_5, and elements of $G.2 \setminus G$ are isometries of the form. In particular, there is an element of order 2 in $G.2 \setminus G$ with character value 0, requiring eight 1's and eight -1's as eigenvalues. Hence matrices inducing this extension have determinant 1, so when r_5 exists over $\mathbb{F}_q$ we have $G.2 < \mathrm{SO}^{+}_{16}(q)$. When $p \equiv \pm 1 \mod 10$, this happens for $q = p$. Otherwise, when $p \equiv \pm 3 \mod 10$, $G.2 \setminus G \subset \mathrm{CGO}^{+}_{16}(p) \setminus \mathrm{GO}^{+}_{16}(p)$. Multiplying elements of $G.2 \setminus G$ by a scalar element of order 4 (if such an element exists) would give us the isoclinic group $G.2^{-}$, but this does not preserve the form since the only scalars to preserve an orthogonal form are $\pm I_{16}$.

The calculations in `2a10d16calc` also construct a matrix $L \in \mathrm{GL}_{16}(\mathbb{Q}(r_5))$ that normalises G, induces the unique nontrivial outer automorphism of G, preserves the form and has determinant and spinor norm 1. Hence, if r_5 exists over $\mathbb{F}_p$ then we have that $G.2 < \Omega$. Otherwise we can rescale L to lie over $\mathrm{GL}_{16}(\mathbb{Q})$ with integral coefficients, which scales the form by $\frac{5}{9}$. Hence by Remark 3.4.7 we have that since L induces an automorphism of order 2, this automorphism is induced by the δ automorphism of Ω. A separate computation also contained in `2a10d16calc` does this for the case $p = 3$.

As there is a single representation, there is a single $\mathrm{Aut}(\Omega)$-class of subgroups.

Lagrange gives us possible abstract containments of $2^{\cdot}A_{10}$ in $2^{\cdot}A_{11}$, A_{17} and A_{18}. We can rule out the latter two cases by examining the maximal subgroups of $2^{\cdot}A_{10}$ - the only subgroup of index less than 18 has index 10 and contains the central element of $2^{\cdot}A_{10}$; hence $2^{\cdot}A_{10}$ acts on the 10 conjugacy classes of this subgroup as

A_{10}. Thus $2^{\cdot}A_{10}$ has no permutation representation on 18 points or fewer, and hence there are no containments involving A_{17} or A_{18}.

There are abstract containments $2^{\cdot}A_{10} < 2^{\cdot}A_{11}$ and $2^{\cdot}A_{10}.2 < 2^{\cdot}A_{11}.2$. In dimension 16 this occurs only in characteristic 11. Since $2^{\cdot}A_{10}$ has no faithful irreducible representations of degree smaller than 16, and all groups we are considering are contained in $\Omega_{16}^{+}(11)$, we have a containment here. Similarly there is a containment of $2^{\cdot}A_{10}.2 < 2^{\cdot}A_{11}.2$, and thus $2^{\cdot}A_{10}$ and $2^{\cdot}A_{10}.2$ are not maximal in any extension of Ω by automorphisms. $\square$

Proposition 3.4.12. *Let $\Omega = \Omega_{16}^{\epsilon}(q)$ with $q = p^e$, let $G = A_{10}$ be an $\mathcal{S}_1$-subgroup of Ω, and let $S = N_{\Omega}(G)$. Then $q = p = 2$, $\epsilon = +$, $S = G.2$, and we have two Ω-classes of subgroups isomorphic to S, with trivial class stabiliser. The group S is $\mathcal{S}_1$-maximal, there is a single* $\mathrm{Aut}(\Omega)$*-class of subgroups S, and for no other q are there $\mathcal{S}_1$-subgroups of $\Omega_{16}^{\pm}(q)$ isomorphic to G.*

Proof. That G only exists in dimension 16 in characteristic 2 is direct from Table 3.2. The rest of the claims (except those regarding maximality) follow from computer calculations in `a10d16f2calc`, making use of details in two version of the online Atlas ([60], [61]).

We have an abstract containment $A_{10} < A_{17}$. However, the characteristic 2 representation of A_{10} takes the value -8 on elements of order 3, whereas the representation of A_{17} is a deleted permutation module, and hence all character values are greater than or equal to -1. Thus the restriction of A_{17} to any subgroup isomorphic to A_{10} will not be the irreducible 16-dimensional character of A_{10}. There are no other possible containments. $\square$

Proposition 3.4.13. *Let $\Omega = \Omega_{16}^{\epsilon}(q)$ with $q = p^e$, let $G = M_{12}$ be an $\mathcal{S}_1$-subgroup of Ω, and let $S = N_{\Omega}(G)$. Then $q = p = 11$, $\epsilon = +$, $S = G$ and we have four Ω-classes of subgroups isomorphic to S, with class stabiliser $\langle \gamma \rangle$. The group S is $\mathcal{S}_1$-maximal, there is a single* $\mathrm{Aut}(\Omega)$*-class of groups S, and for no other q are there $\mathcal{S}_1$-subgroups of $\Omega_{16}^{\pm}(q)$ isomorphic to G.*

Proof. A straightforward computer calculation using Table 3.2 and the group in [60] shows that $\Omega = \Omega_{16}^{+}(11)$. From [32] we see that $G.2 \setminus G$ has character ring contained in $\mathbb{Z}[r_3, r_5]$. Both r_3 and r_5 exist in $\mathbb{F}_{11}$, so we have $G.2 < \mathrm{GO}_{16}^{+}(11)$. Also from the character table, we see that elements of $G.2 \setminus G$ have determinant -1, and hence $G.2 \setminus G \in \mathrm{GO}_{16}^{+}(11) \setminus \mathrm{SO}_{16}^{+}(11)$. Thus we have by Remark 3.4.7 that the class stabiliser is $\langle \gamma \rangle$. There is a single representation, and hence a single $\mathrm{Aut}(\Omega)$-class.

There is an abstract containment $M_{12} < A_{17}$ in characteristic 11. However, the 11-dimensional characters of M_{12} in characteristic 11 are deleted permutation modules obtained from the standard definition of M_{12} acting on a set of 12 points, and the 16-dimensional character of A_{17} is also a deleted permutation module. Hence it is clear that the restriction of the 16-dimensional character of A_{17} to M_{12} is given by one of the 11-dimensional characters plus 5 copies of the trivial character, and there is no containment here. $\square$

Proposition 3.4.14. *Let $\Omega = \Omega_{16}^{\epsilon}(q)$ with $q = p^e$, let $G = 2^{\cdot}\mathrm{Sz}(8)$ be an $\mathcal{S}_1$-subgroup of Ω, and let $S = N_{\Omega}(G)$. Then $q = p = 13$, $\epsilon = +$, $S = G$ and we have eight Ω-classes of subgroups isomorphic to S, with trivial class stabiliser. The group S is $\mathcal{S}_1$-maximal, there is a single* $\mathrm{Aut}(\Omega)$*-class of groups S, and for no other q are there $\mathcal{S}_1$-subgroups of $\Omega_{16}^{\pm}(q)$ isomorphic to G.*

Proof. All claims are straightforward from Table 3.2, since G has a trivial outer automorphism group and we only have a single representation to consider. The result on the characteristic comes from the table, and the fact that y_7 exists over $\mathbb{F}_{13}$.

Lagrange leaves us with A_{17} as the only possible group which may contain $2^{\cdot}\mathrm{Sz}(8)$; however, an analysis of the maximal subgroups of G shows that there are no subgroups of index less than 17, so we cannot have a faithful degree 17 permutation representation of G. Hence G is maximal. $\square$

Proposition 3.4.15. *There are no $\mathcal{S}_1$-maximal subgroups of $\Omega_{16}^{\epsilon}(q)$, or its extensions by automorphisms, with composition factor M_{11}.*

Proof. From Table 3.2 we see that the 16-dimensional irreducible representation of M_{11} is only defined in characteristic 11. There is an abstract containment of M_{11} in M_{12} (M_{11} is a point stabiliser of M_{12} as a permutation on 12 points). From the character tables in [32], the 16-dimensional irreducible representation of M_{12} has character value 0 on elements of order 8, which is also the case for the 16-dimensional irreducible representation of M_{11}, but not for any other 16-dimensional representation of M_{11} consisting of linear combinations of smaller-degree characters. Hence we have a containment here, and in particular the form preserved by the 16-dimensional representations of M_{11} and M_{12} is the same. Since the 16-dimensional representation of M_{11} is scalar-normalising, we are done. $\square$

Proposition 3.4.16. *Let $\Omega = \Omega_{16}^{\epsilon}(q)$ with $q = p^e$, let $G = \mathrm{L}_3(3)$ be an $\mathcal{S}_1$-subgroup of Ω, and let $S = N_{\Omega}(G)$. Then $q = p = 13$, $\epsilon = +$, $S = G$ and we have four Ω-classes of subgroups isomorphic to S, with class stabiliser $\langle\gamma\rangle$. The group S is $\mathcal{S}_1$-maximal, there is a single* $\mathrm{Aut}(\Omega)$*-class of groups S, and for no other q are there $\mathcal{S}_1$-subgroups of $\Omega_{16}^{\pm}(q)$ isomorphic to G.*

Proof. From Table 3.2 we get the result on the characteristic of Ω, from the matrix in [60] a straightforward MAGMA calculation shows that the orthogonal form preserved is of $+$ type, and from [32] we see that the character ring of $G.2$ is $\mathbb{Z}[r_3]$. The irrationality r_3 exists in $\mathbb{F}_{13}$ so that $G.2 < \mathrm{GO}_{16}^{+}(13)$. From the character table we see that elements of $G.2\backslash G$ have determinant -1, so that $G.2\backslash G \subset \mathrm{GO}_{16}^{+}(13)\backslash\mathrm{SO}_{16}^{+}(13)$. Hence by Remark 3.4.7 we have that the class stabiliser is $\langle\gamma\rangle$. There is a single representation, and hence a single $\mathrm{Aut}(\Omega)$-class.

The only possible containment is $\mathrm{L}_3(3) < A_{17}$ in characteristic 13. The 16-dimensional representation of $\mathrm{L}_3(3)$ has character value -2 on elements of order 3 in class $3A$ (in the notation of [32, p. 22]), whereas the 16-dimensional representation of A_{17} is a deleted permutation representation and hence all character values are greater than or equal to -1. Hence there is no possible containment here.

□

Proposition 3.4.17. *There are no $\mathcal{S}_1$-maximal subgroups of $\Omega_{16}^{\epsilon}(q)$, or its extensions by automorphisms, with composition factor $G = \mathrm{L}_2(16)$.*

Proof. $G.4 = \mathrm{P\Gamma L}_2(16)$ has a permutation representation on 17 points, and we can check directly that $G.4 < A_{17}$, so we have an abstract containment. The 16-dimensional representation of $G.4$ is the deleted permutation representation. Hence by Lemma 3.2.22 it follows that $G.4$ preserves an orthogonal plus-type form if 17 is a square modulo p, and an orthogonal minus-type form otherwise. Since the outer automorphism group of $G.4$ is trivial, we have a trivial class stabiliser. We have an abstract containment of $G.4$ inside A_{17}, and a 16-dimensional representation of A_{17} from Proposition 3.4.9. From [12] and [32] we see that there are no faithful representations of $G.4$ of degree smaller than 16, so we have a containment $\mathrm{L}_2(16).4 < A_{17}$. Since the representation of A_{17} is defined whenever the representation of $G.4$ is, the result follows.

□

Proposition 3.4.18. *Let $\Omega = \Omega_{16}^{\epsilon}(q)$ with $q = p^e$, let $G = \mathrm{L}_2(17)_1$ be an $\mathcal{S}_1$-subgroup of Ω and let $S = N_{\Omega}(G)$. Then $q = p \neq 17$ and $S = G$. We have:*

(i) If $p \equiv \pm1, \pm2, \pm4, \pm8 \mod 17$ then $\epsilon = +$, and we have $(p-1,2)^2$ Ω-classes of subgroups isomorphic to S, with class stabiliser $\langle\gamma\rangle$.

(ii) If $p \equiv \pm3, \pm5, \pm6, \pm7 \mod 17$ then $\epsilon = -$, and we have two Ω-classes of subgroups isomorphic to S, with class stabiliser $\langle\gamma\rangle$.

If $p = 3$ then S is not $\mathcal{S}_1$-maximal in Ω or any extension of Ω by automorphisms, whilst if $p \neq 3$ then the group S is $\mathcal{S}_1$-maximal. There is a single $\mathrm{Aut}(\Omega)$*-class of* *subgroups isomorphic to G and the only other situation where there are subgroups of $\Omega_{16}^{\pm}(q)$ isomorphic to G is described in Proposition 3.4.19.*

Proof. Calculations in `l2171d16calc` find a representation of $\mathrm{L}_2(17)_1$ in $\mathrm{GL}_{16}(\mathbb{Q})$ with integral coefficients which preserves a form with determinant 17 times a square; hence the type of form preserved by G depends on whether 17 is a square modulo p, which leads us to a mod 17 dependence on p. Table 3.2 gives us that $q = p \neq 17$. The calculation also shows that $G.2 \setminus G \subset \mathrm{GO}_{16}^{\pm}(q) \setminus \mathrm{SO}_{16}^{\pm}(q)$ when p is odd, and when q is even $G.2 \setminus G \subset \mathrm{SO}_{16}^{+}(2) \setminus \Omega_{16}^{+}(2)$; hence by Remark 3.4.7 the class stabiliser is $\langle \gamma \rangle$.

Lagrange rules out all possible containments except for $\mathrm{L}_2(17) < A_{17}$ or A_{18}. We do not have $\mathrm{L}_2(17) < A_{17}$ as $\mathrm{L}_2(17)$ has no permutation representation on fewer than 18 points; hence the only possible containment is $\mathrm{L}_2(17)_1 < A_{18}$, which can only occur in characteristic 3. Looking at [32] and the standard definition of $\mathrm{L}_2(17)$ as a permutation group on 18 points, we see that the value of the 16-dimensional character $\mathrm{L}_2(17)_1$ on an element g (with order coprime to 3) corresponds with $|\mathrm{Fix}(g)| - 2$, the character value of the 16-dimensional representation of A_{18} in characteristic 3, so we have a containment here. A similar check with $\mathrm{PGL}_2(17) = \mathrm{L}_2(17).2$ shows that we also have a containment $\mathrm{L}_2(17)_1.2 < A_{18}.2$, and hence $\mathrm{L}_2(17)_1$ is not maximal when $p = 3$. □

Proposition 3.4.19. *Let $\Omega = \Omega_{16}^{\epsilon}(q)$ with $q = p^e$, let $G = \mathrm{L}_2(17)_2$ be an $\mathcal{S}_1$-subgroup of Ω, and let $S = N_{\Omega}(G)$. Then $S = G$ and we have class stabiliser $\langle \gamma \rangle$. We have:*

(i) If $p \equiv \pm 1 \mod 9$ and $p \equiv \pm 1, \pm 2, \pm 4, \pm 8 \mod 17$, then $q = p$, $\Omega = \Omega_{16}^{+}(p)$ and there are three $\mathrm{Aut}(\Omega)$-classes of subgroups isomorphic to S, each splitting into four Ω-classes of subgroups isomorphic to S.

(ii) If $p \equiv \pm 2, \pm 4 \mod 9$ and $p \equiv \pm 1, \pm 2, \pm 4, \pm 8 \mod 17$ then $q = p^3$, $\Omega = \Omega_{16}^{+}(p^3)$ and there is a single $\mathrm{Aut}(\Omega)$-class of subgroups isomorphic to S, splitting into $(p-1,2)^2$ Ω-classes of subgroups isomorphic to S.

(iii) If $p \equiv \pm 1 \mod 9$ and $p \equiv \pm 3, \pm 5, \pm 6, \pm 7 \mod 17$ then $q = p$, $\Omega = \Omega_{16}^{-}(p)$ and there are three $\mathrm{Aut}(\Omega)$-classes of subgroups isomorphic to S, each splitting into two Ω-classes of subgroups isomorphic to S.

(iv) If $p \equiv \pm 2, \pm 4 \mod 9$ and $p \equiv \pm 3, \pm 5, \pm 6, \pm 7 \mod 17$ then $q = p^3$, $\Omega = \Omega_{16}^{-}(p^3)$ and there is a single $\mathrm{Aut}(\Omega)$-class of subgroups isomorphic to S, splitting into two Ω-classes of subgroups isomorphic to S.

The group S is $\mathcal{S}_1$-maximal and the only other situation where there are subgroups of $\Omega_{16}^{\pm}(q)$ isomorphic to G is described in Proposition 3.4.18.

Proof. Table 3.2 show that the character ring of $\mathrm{L}_2(17)_2$ involves the irrationality y_9, which gives us that the order of the field depends on $p \mod 9$. The field automorphisms of Ω permute the three representations if there are nontrivial field automorphisms, whilst all other outer automorphisms of Ω fix all three representations, so the result on $\mathrm{Aut}(\Omega)$ follows.

Calculations in `l2172d16calc` find a representation of $\mathrm{L}_2(17)_2$ in $\mathrm{GL}_{16}(\mathbb{Q})$ with integral coefficients which preserves a form with determinant 17 times a square; hence the type of form preserved by G depends on whether 17 is a square modulo p, which leads us to a mod 17 dependence on p. The calculation also shows that $G.2 \setminus G \subset \mathrm{GO}_{16}^{\pm}(q) \setminus \mathrm{SO}_{16}^{\pm}(q)$ when p is odd, and when q is even $G.2 \setminus G \subset \mathrm{SO}_{16}^{+}(8) \setminus \Omega_{16}^{+}(8)$; hence by Remark 3.4.7 the class stabiliser is $\langle\gamma\rangle$.

The existence of the irrationality y_9 in the character ring for $\mathrm{L}_2(17)_2$ but for no other plus-type representations ensures that $\mathrm{L}_2(17)_2$ is $\mathcal{S}_1$-maximal when it is a p-modular reduction of a representation in characteristic 0; i.e. when $p \neq 2$. When $p = 2$, the representation is defined over $\mathbb{F}_8$, and there are no other $\mathcal{S}_1$-candidate subgroups of $\Omega_{16}^{+}(8)$, so $\mathrm{L}_2(17)_2$ is $\mathcal{S}_1$-maximal here also. $\square$

Linear and unitary cases

We now determine the $\mathcal{S}_1$-maximal subgroups of $\mathrm{SL}_{16}^{\pm}(q)$ and its almost simple extensions. (Recall that this notation is a compact way of referring to both linear and unitary groups.) From Table 3.2 the candidates are $\mathrm{L}_3(3)$ ($p \neq 3, 13$), M_{11} ($p \neq 11$), $2^{\cdot}\mathrm{L}_2(31)$ ($p \neq 2, 31$), $4_2^{\cdot}\mathrm{L}_3(4)$ ($p = 3$), M_{12} ($p \neq 3, 11$), $4^{\cdot}M_{22}$ ($p = 7$), A_{11} ($p = 2$), $2^{\cdot}A_{11}$ ($p \neq 2, 11$), A_{12} ($p = 2$) and $2^{\cdot}A_{12}$ ($p = 3$). We will consider these in reverse order.

Proposition 3.4.20. *Let $\Omega = \mathrm{SL}_{16}^{\pm}(q)$ with $q = p^e$, let $G = 2^{\cdot}A_{12}$ be an $\mathcal{S}_1$-subgroup of Ω, and let $S = N_\Omega(G)$. Then $q = p = 3$, $\Omega = \mathrm{SL}_{16}(3)$, $S = G$, and we have two Ω-classes of subgroups isomorphic to S, with class stabiliser $\langle\gamma\rangle$. The group S is $\mathcal{S}_1$-maximal, there is a single $\mathrm{Aut}(\Omega)$-class of subgroups S, and for no other q are there $\mathcal{S}_1$-subgroups of $\mathrm{SL}_{16}^{\pm}(q)$ isomorphic to G.*

Proof. The character ring of the 16-dimensional representation of $2^{\cdot}A_{12}$ in type $\circ$ depends on the existence of the four irrationalities i_2, i_5, b_{11} and b_{35}. All four of these irrationalities are not real but exist over $\mathbb{F}_3$; hence we have that $2^{\cdot}A_{12} < \mathrm{SL}_{16}(3)$. The two representations are interchanged by the outer automorphism of G and also by the duality automorphism γ; hence the nontrivial outer automorphism of G is induced by γ followed by conjugation by some matrix $g \in \mathrm{GL}_{16}(3)$. Computer calculations in `2a12d16f3calc` show that g has determinant 1, so that the class stabiliser is $\langle\gamma\rangle$. The results on the normaliser follows, and the classes are interchanged by δ, where δ has order $(3-1, 16) = 2$. □

Proposition 3.4.21. *There are no $\mathcal{S}_1$-maximal subgroups of $\mathrm{SL}^{\pm}_{16}(q)$ or its extensions by automorphisms isomorphic to A_{11} or $A_{11}.2$.*

Let $\Omega = \mathrm{SL}^{\pm}_{16}(q)$ with $q = p^e$, let $G = A_{12}$ be an $\mathcal{S}_1$-subgroup of Ω, and let $S = N_\Omega(G)$. Then $q = p = 2$, $\Omega = \mathrm{SU}_{16}(2)$, $S = G$, and we have a single Ω-classes of subgroups isomorphic to S, with class stabiliser $\langle\gamma\rangle$. The group S is $\mathcal{S}_1$-maximal, there is a single $\mathrm{Aut}(\Omega)$-class of subgroups S, and for no other q are there $\mathcal{S}_1$-subgroups of $\mathrm{SL}^{\pm}_{16}(q)$ isomorphic to G.

Proof. From Table 3.2, we see that A_{11} only occurs in characteristic 2, and since $b_{11} \notin \mathbb{R}$ and $b_{11} \notin \mathbb{F}_2$, we have that $A_{11} < \mathrm{SU}_{16}(2)$. Further, the two 16-dimensional modules are dual to one another, and hence interchanged by the unique outer automorphism γ of $\mathrm{Out}(\Omega)$. Since the outer automorphism of A_{11} also interchanges the two representations, we have that there is a single Ω-class of groups here, with class stabiliser $\langle\gamma\rangle$.

Similarly for $G = A_{12}$, we see that $G < \mathrm{SU}_{16}(2)$, with γ inducing the unique outer automorphism of A_{12}.

There is clearly an abstract containment $A_{11} < A_{12}$. Denote by ρ the irreducible 16-dimensional character of A_{12}, and (following the notation of [32, p.192]) by Φ_1, Φ_2 and Φ_3 the irreducible characters of dimension 1, 10 and 16 respectively of A_{11}. Then the restriction of ρ to A_{11} must be either $\Phi_2 + 6\Phi_1$ or Φ_3 (or its dual). However the character values on elements of order 11 must involve the irrationality b_{11} whereas the former character is integer-valued; hence we must have a containment of $\mathcal{S}_1$-subgroups here. Since both A_{11} and A_{12} are stabilised by γ, which induces the unique nontrivial outer automorphism in both cases, and the outer automorphism of A_{12} restricts to the outer automorphism of A_{11}, it follows that A_{11} and $A_{11}.2$ are not maximal in any extension of Ω by automorphisms.

□

Proposition 3.4.22. *Let $\Omega = \mathrm{SL}^{\pm}_{16}(q)$ with $q = p^e$, let $G = 2^{\cdot}A_{11}$ be an $\mathcal{S}_1$-subgroup of Ω, and let $S = N_\Omega(G)$. Then $q = p \neq 2, 11$, $S = G$ and we have:*

- *(i) If $p \equiv 1, 3, 4, 5, 9 \mod 11$ then $\Omega = \mathrm{SL}_{16}(p)$, and we have $(q-1, 16)$ Ω-classes of subgroups isomorphic to S, with class stabiliser $\langle\gamma\rangle$.*

(ii) If $p \equiv 2, 6, 7, 8, 10 \mod 11$ then $\Omega = \mathrm{SU}_{16}(p)$ and we have $(q+1, 16)$ Ω-classes of subgroups isomorphic to S, with class stabiliser $\langle \gamma \rangle$.

If $p = 3$, then G and $G.2$ are not $\mathcal{S}_1$-maximal in Ω or any extension of Ω by automorphisms, whilst if $p \neq 3$ then the group S is $\mathcal{S}_1$-maximal. There is a single $\mathrm{Aut}(\Omega)$-class of subgroups isomorphic to S, and for no other q are there $\mathcal{S}_1$-subgroups of $\mathrm{SL}^{\pm}_{16}(q)$ isomorphic to $2^{\cdot}A_{11}$.

Proof. The congruences on p follow from Table 3.2.

In the file `2a11d16calc` we perform calculations in $\mathrm{SL}_{16}(K)$ where $K := \mathbb{Q}(b_{11})$, which show that the outer automorphism of $2^{\cdot}A_{11}$ is induced by γ composed by conjugation with a matrix L with determinant a square. Hence in the linear case it follows from Lemma 3.2.14 that the class stabiliser contains γ. In the unitary case, in the notation of Proposition 3.2.18 we have that $\zeta = 1$ and $\epsilon = 1$, so that again the class stabiliser is γ.

The only possibility for containments is in characteristic 3 with $2^{\cdot}A_{11} < 2^{\cdot}A_{12}$. There is an abstract containment of groups, and looking at the character values we see that on elements of order 11, the 16-dimensional representations of both $2^{\cdot}A_{11}$ and $2^{\cdot}A_{12}$ take character values involving the irrationality b_{11}, whereas all representations of $2^{\cdot}A_{11}$ of smaller degree have integer-valued characters. Hence we have a containment in characteristic 3. Further, both $2^{\cdot}A_{11}$ and $2^{\cdot}A_{12}$ have class stabiliser γ, and the unique nontrivial outer automorphism of $2^{\cdot}A_{12}$ induces the unique nontrivial outer automorphism of $2^{\cdot}A_{11}$, so that $2^{\cdot}A_{11}$ and $2^{\cdot}A_{11}.2$ are not maximal in any extension of Ω by automorphisms. □

Proposition 3.4.23. *Let $\Omega = \mathrm{SL}^{\pm}_{16}(q)$ with $q = p^e$, let $G = 4^{\cdot}M_{22}$ be an $\mathcal{S}_1$-subgroup of Ω and let $S = N_{\Omega}(G)$. Then $q = p = 7$, $\Omega = \mathrm{SU}_{16}(7)$, $S = G$, and we have eight Ω-classes of subgroups isomorphic to S, with class stabiliser $\langle \gamma \rangle$. The group S is $\mathcal{S}_1$-maximal, there is a single $\mathrm{Aut}(\Omega)$-class of subgroups S, and for no other q are there $\mathcal{S}_1$-subgroups of $\mathrm{SL}^{\pm}_{16}(q)$ isomorphic to G.*

Proof. The character ring involves the irrationalities i and b_{11}, neither of which are real, and neither of which exist in $\mathbb{F}_7$; hence by Lemma 3.2.8 we have that $4^{\cdot}M_{22}$ preserves a unitary form. Standard computer calculations found in `4m22d16f7calc` gives the result on the class stabiliser. The Ω-classes are permuted by the δ automorphism of Ω which has order $(16, 7+1) = 8$, and the rest of the results are standard. There are no possible containments. □

Proposition 3.4.24. *There are no $\mathcal{S}_1$-maximal subgroups of $\mathrm{SL}^{\pm}_{16}(q)$ or its extensions by automorphisms isomorphic to M_{11}.*

Let $\Omega = \mathrm{SL}^{\pm}_{16}(q)$ *with* $q = p^e$*, let* $G = M_{12}$ *be an* $\mathcal{S}_1$*-subgroup of* Ω*, and let* $S = N_\Omega(G)$*. Then* $q = p \neq 3, 11$*,* $S = G$ *and we have:*

(i) If $p \equiv 1, 3, 4, 5, 9 \mod 11$ *then* $\Omega = \mathrm{SL}_{16}(p)$*, and we have* $(q-1, 16)$ Ω*-classes of subgroups isomorphic to* S*. If* $p \equiv 1 \mod 6$ *then the class stabiliser is* $\langle\gamma\rangle$*, whilst if* $p \equiv 5 \mod 6$ *then the class stabiliser is* $\langle\gamma\delta\rangle$*.*

(ii) If $p \equiv 2, 6, 7, 8, 10 \mod 11$ *then* $\Omega = \mathrm{SU}_{16}(p)$*, and we have* $(q+1, 16)$ Ω*-classes of subgroups isomorphic to* S*. If* $p \equiv 1 \mod 6$ *then the class stabiliser is* $\langle\gamma\delta\rangle$*, whilst if* $p \equiv 2, 5 \mod 6$ *then the class stabiliser is* $\langle\gamma\rangle$*.*

If $p = 2$ *then* G *is not* $\mathcal{S}_1$*-maximal in* Ω *or any extension of* Ω *by automorphisms, but* $G.2$ *is* $\mathcal{S}_1$*-maximal in* $\Omega.\langle\gamma\rangle$*. If* $p \neq 2$ *then the group* S *is* $\mathcal{S}_1$*-maximal. There is a single* $\mathrm{Aut}(\Omega)$*-class of subgroups isomorphic to* S*, and for no other* q *are there* $\mathcal{S}_1$*-subgroups of* $\mathrm{SL}^{\pm}_{16}(q)$ *isomorphic to* M_{12}*.*

Proof. From Table 3.2 we can see that M_{11} and M_{12} are both defined in the same characteristic, involve the same character ring and have trivial class stabiliser. There is a well-known abstract containment of M_{11} in M_{12}, and looking at the character tables in [12] and [32] shows that the 16-dimensional irreducible representation of M_{12} has character values involving the irrationality b_{11} on elements of order 11, whilst the character table for M_{11} shows that the 16-dimensional irreducible representation also has character values involving b_{11}, and all smaller-dimensional representations have integer-valued characters. Hence the restriction of the 16-dimensional irreducible representation of M_{12} to M_{11} gives the 16-dimensional irreducible representation of M_{11} and hence we have no $\mathcal{S}_1$-maximal subgroups involving M_{11}. Since M_{11} has trivial automorphism group, M_{11} is not maximal in any extension of Ω by automorphisms.

The congruences for M_{12} follow from Table 3.2 and the result on S is clear since M_{12} has trivial class stabiliser.

In the file `m12d16calc` we perform calculations in $\mathrm{SL}_{16}(K)$ where $K := \mathbb{Q}(b_{11})$, which show that the outer automorphism of M_{12} is induced by γ composed with a diagonal automorphism of $\mathrm{SL}_{16}(K)$, which is induced by conjugation by a matrix with determinant a square multiplied by -3. In the linear case, -3 is a square modulo p if and only if $p \equiv 1 \mod 6$; if this is the case then the class stabiliser is $\langle\gamma\rangle$, otherwise it is $\langle\gamma\delta\rangle$. In the unitary case we use Proposition 3.2.18. In the notation of that proposition, $\zeta = -3$, and $\epsilon = 1$ if and only if $p \equiv 1, 2 \mod 6$; hence it follows that the class stabiliser is $\langle\gamma\delta\rangle$ when $p \equiv 1 \mod 6$, and $\langle\gamma\rangle$ when $p \equiv 5 \mod 6$. When $p = 2$, $\mathrm{SU}_{16}(q)$ has no diagonal automorphisms and so the class stabiliser is $\langle\gamma\rangle$. In all cases δ permutes the $(q \mp 1, 16)$ classes.

We next consider containments. Lagrange rules out a number of possibilities, and we cannot have $M_{12} < A_{11}$ as M_{12} has no permutation representation on fewer than 12 points. An analysis of $2^{\cdot}A_{12}$ shows that the only maximal subgroups where containment of M_{12} is not ruled out by Lagrange are $2^{\cdot}M_{12}$ or $2^{\cdot}A_{11}$, neither of which contain M_{12}. Similar analysis of maximal subgroups of $4^{\cdot}M_{22}$ rules out a containment there as well, leaving A_{12} in characteristic 2 as the only possibility for a containment.

We certainly have an abstract containment $M_{12} < A_{12}$ since M_{12} has a permutation representation on 12 points. The 16-dimensional irreducible representation of A_{12} takes values involving b_{11} on elements of order 11, as does the 16-dimensional irreducible representation of M_{12} whilst all smaller-dimensional representations of M_{12} are integer-valued on such elements; hence we have a containment $M_{12} < A_{12}$. However, it follows from a direct computation that $N_{S_{12}}(M_{12}) = M_{12}$, so that the nontrivial outer automorphism of A_{12} does not induce the nontrivial outer automorphism of M_{12}. Since both automorphisms are induced by the γ automorphism of Ω, it follows that we have a type 2 novelty, and $M_{12}.2$ is $\mathcal{S}_1$-maximal in extensions of Ω containing γ. $\square$

Proposition 3.4.25. *Let* $\Omega = \mathrm{SL}^{\pm}_{16}(q)$ *with* $q = p^e$*, let* $G = 4^{\cdot}_2\mathrm{L}_3(4)$ *be an* $\mathcal{S}_1$*-subgroup of* Ω*, and let* $S = N_\Omega(G)$*. Then* $q = p = 3$*,* $\Omega = \mathrm{SU}_{16}(3)$*,* $S = G.2_2$*, and we have four* Ω*-classes of subgroups isomorphic to* S*, with class stabiliser* $\langle\gamma\rangle$*. The group* S *is* $\mathcal{S}_1$*-maximal, there is a single* $\mathrm{Aut}(\Omega)$*-class of subgroups* S*, and for no other* q *are there* $\mathcal{S}_1$*-subgroups of* $\mathrm{SL}^{\pm}_{16}(q)$ *isomorphic to* G*.*

Proof. From the table we obtain that the only possibility is $4^{\cdot}_2\mathrm{L}_3(4) < \mathrm{SU}_{16}(3)$, since both i and b_7 are not real and not realisable over $\mathbb{F}_3$. From [32] we see that the character ring of $4^{\cdot}_2\mathrm{L}_3(4).2_2 \setminus 4^{\cdot}_2\mathrm{L}_3(4)$ contains no additional irrationalities, so that $4^{\cdot}_2\mathrm{L}_3(4).2_2 < \mathrm{GU}_{16}(3)$. From the character tables we can also see that the element in class 2C has trace 4, and must square to an element of trace 16. Hence it must have 1 (10 times) and -1 (6 times) as its eigenvalues, and hence it has determinant 1. Thus $4^{\cdot}_2\mathrm{L}_3(4).2_2 < \mathrm{SU}_{16}(3)$, and so $N_\Omega = G.2_2$. Computer calculations found in `42l34d16f3calc` confirm that γ stabilises the class, and hence induces the unique outer automorphism of S, while δ permutes the classes.

Lagrange limits the possibility of abstract containments to $2^{\cdot}A_{12}$, and an analysis of the maximal subgroups of $2^{\cdot}A_{12}$ confirms that none of them can contain $4^{\cdot}_2\mathrm{L}_3(4).2_2$, so there are no containments in this case. $\square$

Proposition 3.4.26. *Let* $\Omega = \mathrm{SL}^{\pm}_{16}(q)$ *with* $q = p^e$*, let* $G = 2^{\cdot}\mathrm{L}_2(31)$ *be an* $\mathcal{S}_1$*-subgroup of* Ω*, and let* $S = N_\Omega(G)$*. Then* $q = p \neq 2, 31$*,* $S = G$ *and we have:*

(i) If $p \equiv 1,2,4,5,7,8,9,10,14,16,18,19,20,25,28 \mod 31$ *then we have* $\Omega = \mathrm{SL}_{16}(p)$ *and we have* $(q-1,16)$ Ω*-classes of subgroups isomorphic to* S.

(ii) If $p \equiv 3,6,11,12,13,15,17,21,22,23,24,26,27,29,30 \mod 31$ *then we have* $\Omega = \mathrm{SU}_{16}(p)$ *and we have* $(q+1,16)$ Ω*-classes of subgroups isomorphic to* S.

If $p \equiv \pm 1 \mod 8$ *then the class stabiliser is* $\langle \gamma \rangle$*; if* $p \equiv \pm 3 \mod 8$ *then the class stabiliser is* $\langle \gamma\delta \rangle$*. The group* S *is* $\mathcal{S}_1$*-maximal, there is a single* $\mathrm{Aut}(\Omega)$*-class of groups* S*, and for no other* q *are there* $\mathcal{S}_1$*-subgroups of* $\mathrm{SL}^{\pm}_{16}(q)$ *isomorphic to* $2^{\cdot}\mathrm{L}_2(31)$.

Proof. The congruences on p for Ω follow directly from Table 3.2.

In the file `2l231d16calc` we perform calculations in $\mathrm{SL}_{16}(K)$ where $K := \mathbb{Q}(b_{31})$, which show that the outer automorphism of $2^{\cdot}\mathrm{L}_2(31)$ is induced by γ composed with a diagonal automorphism of $\mathrm{SL}_{16}(K)$ with determinant twice a square. In the linear case, 2 is a square modulo p if and only if $p \equiv \pm 1 \mod 8$; hence if this is the case then the class stabiliser is $\langle \gamma \rangle$, otherwise it is $\langle \gamma\delta \rangle$. A similar calculation applies in the unitary case to give the same result via Proposition 3.2.18. In all cases δ permutes the $(q \mp 1, 16)$ classes.

$2^{\cdot}\mathrm{L}_2(31)$ is the only $\mathcal{S}_1$-candidate with order divisible by 31 so this is $\mathcal{S}_1$-maximal. □

Proposition 3.4.27. *Let* $\Omega = \mathrm{SL}^{\pm}_{16}(q)$ *with* $q = p^e$*, let* $G = \mathrm{L}_3(3)$ *be an* $\mathcal{S}_1$*-subgroup of* Ω*, and let* $S = N_{\Omega}(G)$*. Then* $S = G$ *and we have:*

(i) If $p \equiv 1,3,9 \mod 13$ *and* $p \neq 3$ *then* $q = p$*,* $\Omega = \mathrm{SL}_{16}(p)$*, and there are* $(q-1,16)$ Ω*-classes of subgroups isomorphic to* S*. If* $p \equiv 1 \mod 6$ *then the class stabiliser is* $\langle \gamma \rangle$*; if* $p \equiv 5 \mod 6$ *then the class stabiliser is* $\langle \gamma\delta \rangle$*. There are two* $\mathrm{Aut}(\Omega)$*-classes of such groups* S.

(ii) If $p \equiv 4,10,12 \mod 13$ *then* $q = p$*,* $\Omega = \mathrm{SU}_{16}(p)$*, and there are* $(q+1,16)$ Ω*-classes of subgroups isomorphic to* S*. If* $p \equiv 1 \mod 6$ *then the class stabiliser is* $\langle \gamma\delta \rangle$*; if* $p \equiv 5 \mod 6$ *then the class stabiliser is* $\langle \gamma \rangle$*. There are two* $\mathrm{Aut}(\Omega)$*-classes of such groups* S.

(iii) If $p \equiv 2,5,6,7,8,11 \mod 13$ *then* $q = p^2$*,* $\Omega = \mathrm{SU}_{16}(p^2)$*, and there are* $2(q+1,16)$ Ω*-classes of subgroups isomorphic to* S*, with class stabiliser* $\langle \gamma\delta \rangle$*. There is a single* $\mathrm{Aut}(\Omega)$*-class of such groups* G.

The group S *is* $\mathcal{S}_1$*-maximal, and for no other* q *are there* $\mathcal{S}_1$*-subgroups of* $\mathrm{SL}^{\pm}_{16}(q)$ *isomorphic to* $\mathrm{L}_3(3)$.

Proof. The congruences on p for Ω follow directly from Table 3.2.

We have a single nontrivial outer automorphism of G, but four representations, upon which this automorphism acts as a (2,2)-cycle. Hence, we have two orbits of the action of the conformal group $\mathrm{CGL}^{\pm}_{16}(q)$ on the representations of G. Standard computations confirm that each of these orbits splits into d Ω-classes, with $d = |\delta|$, giving us $2d$ Ω-classes in total.

In the file `l33d16calc` we perform calculations in $\mathrm{SL}_{16}(K)$ where $K := \mathbb{Q}(d_{13})$, which show that the outer automorphism of $\mathrm{L}_3(3)$ is induced by γ composed with a diagonal automorphism of $\mathrm{SL}_{16}(K)$ with determinant a square multiplied by -3. Thus we obtain the class stabilisers with a similar argument to the proof of Proposition 3.4.24. In particular, note that since -3 is always a square in $\mathbb{F}_{p^2}$ it follows that when $\Omega = \mathrm{SU}_{16}(p^2)$ the class stabiliser is always $\gamma\delta$.

In the case where $q = p^2$, so the base field is $\mathbb{F}_{p^4}$, we have ϕ which acts as the p-power map on character values. However from [12] we see that the nontrivial outer automorphism of G acts on character values as the p^2-power map on character values, so that no outer automorphism of G acts as ϕ on the character values. Hence we have a single $\mathrm{Aut}(\Omega)$-class here, and so again the computation in the previous paragraph suffices to confirm that the class stabiliser is $\langle\gamma\rangle$ here also.

When $p = 2$, $\Omega = \mathrm{SU}_{16}(2)$ has only a single nontrivial outer automorphism, so the outer automorphism of G is induced by γ.

$\mathrm{L}_3(3)$ is the only $\mathcal{S}_2$-candidate with order divisible by 13 so this is $\mathcal{S}_1$-maximal. □

3.4.3 Summary

Theorem 3.4.28. *Let G and Ω be as in Remark 3.3.12 with $\Omega = \mathrm{Sp}_{16}(q)$. Then representatives of the conjugacy classes of $\mathcal{S}_1$-maximal subgroups of G are described in the list below:*

Proof. See discussion in Section 3.4.2. □

(i) $S = A_{18}.2 < \mathrm{Sp}_{16}(2)$ with trivial class stabiliser.

(ii) $S = 2^{\cdot}A_8 < \mathrm{Sp}_{16}(7)$ with class stabiliser $\langle\delta\rangle$.

(iii) $S = 2^{\cdot}\mathrm{L}_2(17)$ if $p \equiv \pm 5 \mod 12$ and $p \neq 17$, and $S = 2^{\cdot}\mathrm{L}_2(17).2$ with trivial class stabiliser if $p \equiv \pm 1 \mod 12$, $S < \mathrm{Sp}_{16}(p)$ in both cases, and the class stabiliser is $\langle\delta\rangle$ in the first case and trivial in the second.

(iv) $S = 2^{\cdot}\mathrm{L}_2(17).2$ if $p \equiv 1, 11, 13, 23, 25, 35 \mod 36$, and $S = 2^{\cdot}\mathrm{L}_2(17)$ otherwise, provided $p \neq 2, 3, 17$. The class stabiliser is trivial in the first case and $\langle\delta\rangle$ in the second. If $p \equiv \pm 1 \mod 9$ then $S < \mathrm{Sp}_{16}(p)$; otherwise $S < \mathrm{Sp}_{16}(p^3)$.

Theorem 3.4.29. *Let G and Ω be as in Remark 3.3.12 with $\Omega = \Omega^+_{16}(q)$. Then representatives of the conjugacy classes of $\mathcal{S}_1$-maximal subgroups of G are described in the list below:*

Proof. See discussion in Section 3.4.2. □

(i) $S = A_{17}$ with $q = p \equiv \pm 1, \pm 2, \pm 4, \pm 8 \mod 17$ and class stabiliser $\langle\gamma\rangle$.

(ii) $S = 2^{\cdot}A_{11}.2 < \Omega^+_{16}(11)$ with trivial class stabiliser.

(iii) $S^\infty = 2^{\cdot}A_{10}$ with $q = p$. If $p \equiv \pm 3 \mod 10$ then $S = 2^{\cdot}A_{10}$ with class stabiliser $\langle\delta\rangle$; whereas if $p \equiv \pm 1 \mod 10$ and $p \neq 11$, then $S = 2^{\cdot}A_{10}.2$ with trivial class stabiliser.

(iv) $S = A_{10}.2 < \Omega^+_{16}(2)$ with trivial class stabiliser.

(v) $S = M_{12} < \Omega^+_{16}(11)$ with class stabiliser $\langle\gamma\rangle$.

(vi) $S = 2^{\cdot}\mathrm{Sz}(8) < \Omega^+_{16}(13)$ with trivial class stabiliser.

(vii) $S = \mathrm{L}_3(3) < \Omega^+_{16}(13)$ with class stabiliser $\langle\gamma\rangle$.

(viii) $S = \mathrm{L}_2(17)$ with $q = p \equiv \pm 1, \pm 2, \pm 4, \pm 8 \mod 17$ and class stabiliser $\langle\gamma\rangle$.

(ix) $S = \mathrm{L}_2(17)$ with $p \equiv \pm 1, \pm 2, \pm 4, \pm 8 \mod 17$ and class stabiliser $\langle\gamma\rangle$. If $p \equiv \pm 1 \mod 9$ then $S < \Omega^+(p)$, and if $p \equiv \pm 2, \pm 4 \mod 9$ then $S < \Omega^+(p^3)$.

Theorem 3.4.30. *Let G and Ω be as in Remark 3.3.12 with $\Omega = \Omega^-_{16}(q)$. Then representatives of the conjugacy classes of $\mathcal{S}_1$-maximal subgroups of G are described in the list below:*

Proof. See discussion in Section 3.4.2. □

(i) $S = A_{18} < \Omega^-_{16}(3)$ with class stabiliser $\langle\gamma\rangle$.

(ii) $S = A_{17} < \Omega^-_{16}(p)$ with $p \equiv \pm 3, \pm 5, \pm 6, \pm 7 \mod 17$, $p \neq 3$ and class stabiliser $\langle\gamma\rangle$.

(iii) $S = \mathrm{L}_2(17) < \Omega^-_{16}(p)$ with $p \equiv \pm 3, \pm 5, \pm 6, \pm 7 \mod 17$, $p \neq 3$ and class stabiliser $\langle\gamma\rangle$.

(iv) $S = \mathrm{L}_2(17)$ with $p \equiv \pm 3, \pm 5, \pm 6, \pm 7 \mod 17$ and class stabiliser $\langle\gamma\rangle$. If $p \equiv \pm 1 \mod 9$ then $S < \Omega_{16}^{-}(p)$, and if $p \equiv \pm 2, \pm 4 \mod 9$ then $S < \Omega_{16}^{-}(p^3)$.

Theorem 3.4.31. *Let G and Ω be as in Remark 3.3.12 with $\Omega = \mathrm{SL}_{16}(q)$ or $\mathrm{SU}_{16}(q)$. Then representatives of the conjugacy classes of $\mathcal{S}_1$-maximal subgroups of G are described in the list below:*

Proof. See discussion in Section 3.4.2. □

(i) $S = 2^{\cdot}A_{12} < \mathrm{SL}_{16}(3)$ with class stabiliser $\langle\gamma\rangle$.

(ii) $S = A_{12} < \mathrm{SU}_{16}(2)$ with class stabiliser $\langle\gamma\rangle$.

(iii) $S = 2^{\cdot}A_{11}$ with $p \neq 2, 3, 11$ with class stabiliser $\langle\gamma\rangle$. If $p \equiv 1, 3, 4, 5, 9 \mod 11$ then $S < \mathrm{SL}_{16}(p)$ and if $p \equiv 2, 6, 7, 8, 10 \mod 11$ then $S < \mathrm{SU}_{16}(p)$.

(iv) $S = 4^{\cdot}M_{22} < \mathrm{SU}_{16}(7)$ with class stabiliser $\langle\gamma\rangle$.

(v) $S = M_{12}$ with $p \neq 3, 11$. If $p \equiv 1, 3, 4, 5, 9 \mod 11$ then $S < \mathrm{SL}_{16}(p)$, with class stabiliser $\langle\gamma\rangle$ if $p \equiv 1 \mod 6$ or $\langle\gamma\delta\rangle$ if $p \equiv 5 \mod 6$. If $p \equiv 2, 6, 7, 8, 10 \mod 11$ and $p \neq 2$ then $S < \mathrm{SU}_{16}(p)$ with class stabiliser $\langle\gamma\delta\rangle$ if $p \equiv 1 \mod 6$ and $\langle\gamma\rangle$ if $p \equiv 5 \mod 6$. If $p = 2$ then S is not $\mathcal{S}_1$-maximal in Ω, but $S.2$ is $\mathcal{S}_1$-maximal in $\Omega.\gamma$.

(vi) $S = 4_2^{\cdot}\mathrm{L}_3(4).2_2 < \mathrm{SU}_{16}(3)$ with class stabiliser $\langle\gamma\rangle$.

(vii) $S = 2^{\cdot}\mathrm{L}_2(31)$ with $p \neq 2, 31$. If p is a square modulo 31 then $S < \mathrm{SL}_{16}(p)$, whilst if p is not a square modulo 31 then $S < \mathrm{SU}_{16}(p)$. We have class stabiliser $\langle\gamma\rangle$ if $p \equiv \pm 1 \mod 8$ and $\langle\gamma\delta\rangle$ if $p \equiv \pm 3 \mod 8$.

(viii) $S = \mathrm{L}_3(3)$ with $p \neq 3, 13$. If $p \equiv 1, 3, 9 \mod 13$ then $S < \mathrm{SL}_{16}(p)$, with class stabiliser $\langle\gamma\rangle$ if $p \equiv 1 \mod 6$ and $\langle\gamma\delta\rangle$ if $p \equiv 5 \mod 6$. If $p \equiv 4, 10, 12 \mod 13$ then $S < \mathrm{SU}_{16}(p)$, with class stabiliser $\langle\gamma\delta\rangle$ if $p \equiv 1 \mod 6$ and $\langle\gamma\rangle$ if $p \equiv 5 \mod 6$. If $p \equiv 2, 5, 6, 7, 8, 11 \mod 13$ then $S < \mathrm{SU}_{16}(p^2)$, with class stabiliser $\langle\gamma\delta\rangle$ for all such p.

Chapter 4

$\mathcal{S}_2$-candidates

4.1 Introduction

In this chapter, we will classify the $\mathcal{S}_2$-candidates in dimensions 16 and 17. We begin with an introduction to the theory of highest weights, including results we will use in the remaining sections. We will also define a subclass $\mathcal{S}_2^*$ of $\mathcal{S}_2$ which consists of groups which are not obviously contained in a group in class $\mathcal{C}_4$ or $\mathcal{C}_7$.

We will then follow a similar procedure to Chapter 3 for the rest of the chapter. Section 4.2 lists the $\mathcal{S}_2^*$-candidate subgroups, which are described in more detail in Section 4.3, including computations of the minimal fields of realisations, forms preserved and class stabilisers inside the conformal group. Section 4.4 determines the action of graph and field automorphisms on $\mathcal{S}_2^*$-candidate subgroups, and Section 4.5 analyses containments between them. Finally, Section 4.6 summarises the results of this chapter.

We remind the reader that the Magma computations referenced here can be found at `https://github.com/danielrogerswarwick/thesis`.

4.1.1 Highest Weight Theory

In this section we introduce the aspects of highest weight theory we will need for some of the later computations. We will follow a similar approach to [48, Section 8.1], which is in turn based on the comprehensive treatment in [43]. In particular, many of the definitions in this section will be specific cases of the more general definitions as given in [43].

Throughout this section, let G denote a classical group over an algebraically closed field K; we will typically take $K = \overline{\mathbb{F}}_p$. It follows from [43, Section 1.2] that such groups are examples of algebraic groups. We will not define an algebraic group, but will apply the theory of algebraic groups to the classical groups we are interested in.

Definition 4.1.1. (i) The group of diagonal matrices of $\mathrm{GL}_l(K)$ is denoted $D_l(K)$.

(ii) A subgroup T of G is a *torus* if it is isomorphic to $D_l(K)$ for some l.

(iii) The group T is a *maximal torus* if it is maximal among the tori of G with respect to inclusion.

s

Example 4.1.2. A maximal torus T of $\mathrm{GL}_n(K)$ is $D_n(K)$. A maximal torus of $\mathrm{SL}_n(K)$ is $T \cap \mathrm{SL}_n(K) \cong D_{n-1}(K)$.

Definition 4.1.3. Let T be a torus of G, where we identify T with $D_l(K) = \langle \mathrm{diag}(t_1, \dots, t_l) : t_i \in K^* \rangle$.

(i) A *character* of a maximal torus T of G is a map $T \to K^*$, given by

$$\mathrm{diag}(t_1, \dots, t_l) \mapsto t_1^{x_1} \dots t_l^{x_l}$$

for some $x_i \in \mathbb{Z}$. The $\mathbb{Z}$-module of all characters is denoted $X(T)$.

(ii) A *cocharacter* of a maximal torus T of G is a map $K^* \to T$ given by $t \mapsto \mathrm{diag}(t^{y_1}, \dots, t^{y_l})$ for some $y_i \in \mathbb{Z}$. The $\mathbb{Z}$-module of all cocharacters is denoted $Y(T)$.

Remark 4.1.4. It follows from [43, Example 3.5] that for $\chi \in X(T)$, $\gamma \in Y(T)$ and $t \in K^*$, $t\gamma\chi$ will be an integral power of t that does not depend on t. We denote the coefficient by $\langle \chi, \gamma \rangle \in \mathbb{Z}$, so that $t\gamma\chi = t^{\langle \chi, \gamma \rangle}$.

Definition 4.1.5. Let T be a maximal torus of G. Then the *set of roots* of G is $\Phi(G)$, a certain subset of the set of characters $X(T)$. The exact definition relies on theory relating to the underlying Lie algebra; see for instance [9, Chapter 3] for details.

A set $\Delta \subset \Phi(G)$ is a *base* for $\Phi(G)$ if it is a basis of $\Phi(G)$ viewed as a $\mathbb{Z}$-module, and additionally every $\beta \in \Phi(G)$ is expressible as $\sum\limits_{\alpha \in \Phi(G)} c_\alpha \alpha$ with either all $c_\alpha \geq 0$ or all $c_\alpha \leq 0$.

To each root α we can define a corresponding *coroot* $\overset{\vee}{\alpha} \in Y(T)$. (See [43, Lemma 8.19] for the precise definition).

Definition 4.1.6. Let V be a finite-dimensional vector space over $\overline{\mathbb{F}}_p$, and let $\rho : G \to \mathrm{GL}(V)$ be a representaton which is also a morphism of algebraic groups (see [43, p. 3] for a definition). Then ρ is said to be a *rational representation* of G.

Definition 4.1.7. Let $\rho : G \to \mathrm{GL}(V)$ be a rational representation of G, and $T < G$ be a maximal torus. Then for $\lambda \in X(T)$ we define

$$V_\lambda = \{v \in V | v(t\rho) = v(t\lambda)\ \forall t \in T\}.$$

If $V_\lambda \neq 0$, then we say that λ is a *weight* of V.

As in [48, Definition 8.1.11], we make a specific definition of a Borel subgroup, rather than the more general definition given in [43, Definition 6.3]; for a justification of the following definition, see [43, Example 6.7].

Definition 4.1.8. The subgroup of G consisting of all lower-triangular matrices in G is called the *Borel subgroup* of G.

Lemma 4.1.9. *[43, Theorem 4.1] Let B be the Borel subgroup of G, and $\rho : G \to \mathrm{GL}(V)$ be a rational representation. Then there exists a vector $v^+ \in V$ such that the 1-dimensional vector space generated by v^+ is invariant under $B\rho$.*

Definition 4.1.10. Let $v^+ \in V \setminus 0$ be a vector such that $\langle v^+ \rangle$ is invariant under $B\rho$. Then v^+ is a *maximal vector* of V.

Note that for every maximal vector $v^+ \in V$, there always exists a weight $\lambda \in X(T)$ such that $v^+ \in V_\lambda$.

Definition 4.1.11. Let $\rho : G \to \mathrm{GL}(V)$ be a representation such that ρ maps the Borel subgroup of G to the Borel subgroup of $G\rho$, and a maximal torus of G to a maximal torus of $G\rho$. (In particular, ρ maps lower triangular matrices to lower triangular matrices). Then ρ is said to be *lower-triangular-preserving.*

Note that many of the representations we will consider, such as the tensor product and symmetric and exterior powers, are lower-triangular-preserving.

Remark 4.1.12. It is clear that if ρ is lower-triangular-preserving, then a maximal vector of V is $v^+ = (1, 0, \dots, 0)$. In this case, $v^+(t\rho) = (t\rho)_{1,1}v^+ = (t\lambda)v^+$, and thus $t\lambda$ can be determined from the first entry of the diagonal matrix $t\rho$.

Definition 4.1.13. Let $\rho : G \to \mathrm{GL}(V)$ be a rational representation, and suppose V is generated as a $G\rho$-module by a maximal vector v^+. Let $\lambda \in X(T)$ be a weight such that $v^+ \in V_\lambda$. Then V is called a *highest weight module* of G, and λ is called the *highest weight.*

Definition 4.1.14.

(i) Let $\Delta = \{\alpha_1, \ldots, \alpha_m\}$ denote a base of the root system Φ of G with respect to the maximal torus T. We say that $\lambda_i \in X(T)$ is a *fundamental dominant weight* if $\langle \lambda_i, \overset{\vee}{\alpha}_j \rangle = \delta_{i,j}$ for all j, where $\delta_{i,j}$ is the Kronecker delta.

(ii) The character $\lambda \in X(T)$ is *dominant* if we can express it as an integral sum $\sum_{i=1}^{m} a_i \lambda_i$ of fundamental dominant weights with nonnegative coefficients a_i.

We often identify dominant weights with the list of nonnegative integers $(a_1, \ldots, a_m)$. For instance this notation is used in [41].

Lemma 4.1.15. *[43, Proposition 15.9] Let $\rho : G \to \mathrm{GL}(V)$ be a rational representation. Let v^+ be a maximal vector of V with associated weight $\lambda \in X(T)$. Then λ is dominant.*

Remark 4.1.16. Hence, we will often refer to a representation ρ having highest weight $(a_1, \ldots, a_m)$.

Lemma 4.1.17. *Let $\rho : G \to \mathrm{GL}(V)$ be a lower-triangular-preserving rational representation with corresponding highest weight λ. Let T denote a torus of G, let Δ be a base for G and let $\overset{\vee}{\alpha}_i : K^* \to T$ be the coroot associated with $\alpha_i \in \Delta$. Then ρ has highest weight $(a_1, \ldots, a_m)$, where a_i is a nonnegative integer such that for every $a \in K^*$, $(a\overset{\vee}{\alpha}_i\rho)_{1,1} = a^{a_i}$.*

Proof. Since ρ is lower-triangular-preserving, it follows that $T\rho$ is a maximal torus of $G\rho$, and so the maps $\overset{\vee}{\alpha}_i\rho : K \to G\rho$ lies in $Y(T\rho)$, and thus can be taken to be coroots corresponding to $G\rho$. Since λ is dominant by Lemma 4.1.15, we can write $\lambda = \sum_{i=1}^{m} a_i\lambda_i$ for some non-negative a_i, where the λ_i are fundamental dominant weights. Then by linearity of $\langle\ ,\ \rangle$ we have that $\langle \lambda, \overset{\vee}{\alpha}_i\rho \rangle = a_i$.

Since ρ is lower-triangular-preserving, we also have from Remark 4.1.12 that $a\overset{\vee}{\alpha}_i\rho\lambda = (a\overset{\vee}{\alpha}_i\rho)_{1,1}$. However, by definition we also have $a\overset{\vee}{\alpha}_i\rho\lambda = a^{\langle \lambda, \overset{\vee}{\alpha}_i\rho \rangle} = a^{a_i}$, so that a_i is equal to the power of a in $(a\overset{\vee}{\alpha}_i\rho)_{1,1}$. $\square$

Thus, in many of the cases we will be considering, we can determine the weight of a representation from the first entries of diagonal matrices.

Theorem 4.1.18. *[43, Theorem 15.17]*

(i) Two irreducible rational representations ρ_1 and ρ_2 of G with corresponding highest weights λ_1 and λ_2 are equivalent if and only if $\lambda_1 = \lambda_2$.

(ii) If λ is a dominant weight of a rational representation ρ then there exists an irreducible rational representation of G with highest weight λ.

Hence there is a one-to-one correspondence between irreducible rational representations of G and dominant weights.

4.1.2 The Steinberg theorems

The fields discussed in the results of the previous section are all algebraically closed. In this section, we will describe the theory of irreducible $\overline{\mathbb{F}}_p$-representations of groups of Lie type over $\overline{\mathbb{F}}_p$, and how this theory translates to finite fields. We follow closely the approach of [8, Section 5.1.1].

Let ${}^tX_l(q)$ denote a simple group of Lie type with Lie rank l, and ${}^t\hat{X}_l(q)$ denote the simply connected version of ${}^tX_l(q)$ obtained by extending ${}^tX_l(q)$ by the part of the Schur multiplier coprime to p. It is a well-known result (see for instance [36, Section V.5]) that $\overline{\mathbb{F}}_p$ is unique up to isomorphism and contains $\mathbb{F}_{p^e}$ for every $e \geq 1$. In particular this allows us to realise the group ${}^t\hat{X}_l(p^e)$ as a subgroup of $\hat{X}_l(\overline{\mathbb{F}}_p)$. For untwisted groups (i.e. groups with $t = 1$) this is obtained by restricting the field (in other words by considering the centraliser of the field automorphism ϕ^e), whereas for twisted groups we consider the centraliser of $\gamma^{-1}\phi^e$ instead, where γ is the graph automorphism of order t inducing the twist.

Recall Definition 4.1.7 of a weight, and how to each weight λ we can assign an element of $\mathbb{N}^l$. As a consequence of Theorem 4.1.18, given a weight $\lambda \in \mathbb{N}^l$ associated with $\hat{X}_l(\overline{\mathbb{F}}_p)$ we can uniquely identify a corresponding module whose highest weight is λ, which we will denote $M(\lambda)$.

Definition 4.1.19. A weight $\lambda = (a_1, \ldots, a_l)$ is *m-restricted* if $0 \leq a_i \leq m-1$ for all i.

Let $M(\lambda)$ be an $\hat{X}_l(\overline{\mathbb{F}}_p)$-module, with associated representation $\rho : \hat{X}_l(\overline{\mathbb{F}}_p) \to \mathrm{GL}(V)$. Recall that by ${}^\phi M(\lambda)$, we mean the application of the field automorphism ϕ of $\hat{X}_l(\overline{\mathbb{F}}_p)$ to $M(\lambda)$, and by $M(\lambda)^\phi$ we mean the application of the field automorphism of $\mathrm{GL}(V)$ to the image of ρ.

Theorem 4.1.20 (Steinberg's Tensor Product Theorem)**.** *[50, Theorem 5.1] Let $\lambda_0, \ldots, \lambda_r$ be p-restricted weights associated with $\hat{X}_l(\overline{\mathbb{F}}_p)$, and let ϕ be the Frobenius automorphism of $\hat{X}_l(\overline{\mathbb{F}}_p)$. Then as $\hat{X}_l(\overline{\mathbb{F}}_p)$-modules*

$$M(\lambda_0 + p\lambda_1 + \ldots + p^r\lambda_r) \cong M(\lambda_0) \otimes {}^{\phi}M(\lambda_1) \otimes \ldots \otimes {}^{\phi^r}M(\lambda_r).$$

Thus, to understand highest weight representations, it suffices to understand the p-restricted highest weight representations.

Lemma 4.1.21. *[8, Lemma 5.1.11] Let λ and μ be weights associated with $\hat{X}_l(\overline{\mathbb{F}}_p)$. Then $M(\lambda+\mu)$ occurs as a constituent of $M(\lambda)\otimes M(\mu)$ of multiplicity 1.*

Next, we consider what happens if we restrict the representation of $\hat{X}_l(\overline{\mathbb{F}}_p)$ to a finite subgroup $\hat{X}_l(q)$.

Theorem 4.1.22. *[50, Theorem 1.3] and [8, Theorem 5.1.3]. Let ${}^tX_l(q)$ be other than ${}^2\mathfrak{B}_2(q)$, ${}^2\mathfrak{G}_2(q)$ or ${}^2\mathfrak{F}_4(q)$. Then any irreducible $\overline{\mathbb{F}}_p$-module for ${}^t\hat{X}_l(q)$ is isomorphic to the restriction of the $\hat{X}_l(\overline{\mathbb{F}}_p)$-module $M(\lambda)$ to ${}^t\hat{X}_l(q)$ for some q-restricted weight λ. Further, if λ and μ are q-restricted weights, then the restrictions of the $\hat{X}_l(\overline{\mathbb{F}}_p)$-modules $M(\lambda)$ and $M(\mu)$ to $\hat{X}_l(q)$ are isomorphic if and only if $\lambda=\mu$.*

Theorem 4.1.23 (Steinberg's Twisted Tensor Product Theorem)**.** *[50, Theorem 12.2] Let $q=p^e$, and ${}^tX_l(q)$ be other than ${}^2\mathfrak{B}_2(q)$, ${}^2\mathfrak{G}_2(q)$ or ${}^2\mathfrak{F}_4(q)$. Then any irreducible module for ${}^t\hat{X}_l(q)$ over $\overline{\mathbb{F}}_p$ has the form*

$$M_0\otimes M_1^{\phi}\otimes\cdots\otimes M_{e-1}^{\phi^{e-1}}$$

where each M_i is a p-restricted irreducible module, arising from the restriction to ${}^t\hat{X}_l(q)$ of a module $M(\lambda)$ of $\hat{X}_l(\overline{\mathbb{F}}_p)$ with p-restricted weight λ. There are q^l such modules, and these modules are pairwise non-isomorphic.

In this thesis we will not need to consider the groups ${}^2\mathfrak{G}_2(q)$ or ${}^2\mathfrak{F}_4(q)$, since by [41] these do not have 16- or 17-dimensional representations (although there is a similar result to Theorem 4.1.23 for these groups). The groups ${}^2\mathfrak{B}_2(q)$ only occur when q is even. These groups are known as the Suzuki groups; see for example [60, Section 4.2] for an introduction to these groups.

Theorem 4.1.24. *[50, Theorem 12.2] Let $G=\mathrm{Sz}(2^{2e+1})$ with natural module V. Then any irreducible module of G over $\overline{\mathbb{F}}_2$ has the form*

$$M_0\otimes M_1^{\phi}\otimes\cdots\otimes M_{2e}^{\phi^{2e}}$$

where $M_i\in\{\mathbb{1},V\}$, and these 2^{2e+1} modules are pairwise nonisomorphic.

We next determine the action of various automorphisms of ${}^t\hat{X}_l(q)$ on irreducible modules of ${}^t\hat{X}_l(q)$.

Proposition 4.1.25. *([8, Proposition 5.1.9] and [33, Proposition 5.4.2]) Let ${}^tX_l(q)$ be other than ${}^2\mathfrak{B}_2(q)$, ${}^2\mathfrak{G}_2(q)$ or ${}^2\mathfrak{F}_4(q)$, and let $M(\lambda)$ be an irreducible module of ${}^tX_l(q)$ with highest weight λ.*

(i) Let δ denote a diagonal automorphism of ${}^tX_l(q)$. Then δ stabilises $M(\lambda)$.

(ii) Let ϕ_X denote the Frobenius automorphism of ${}^tX_l(q)$. Then ${}^{\phi_X}M(\lambda) = M(\lambda)^{\phi_G}$, where ϕ_G is the corresponding Frobenius automorphism of $\mathrm{GL}_n(q^t)$.

(iii) Let γ denote a graph automorphism of an untwisted group (so $t = 1$ and X_l is one of $\mathfrak{A}_l$, $\mathfrak{D}_l$ or $\mathfrak{E}_6$). Then γ acts on the underlying Dynkin diagram via a permutation $s_\gamma \in \mathrm{Sym}(l)$, and ${}^\gamma M(\lambda) = M(\lambda^{s_\gamma})$. In particular, if $X_l = \mathfrak{A}_l$, then the unique nontrivial graph automorphism induces duality on the module.

(iv) Let σ denote the field automorphism ϕ_X^e of a twisted group (so $t \neq 1$), and γ denote the graph automorphism such that ${}^tX_l(q)$ is the centraliser of $\gamma^{-1}\sigma$ in $X_l(\overline{\mathbb{F}}_p)$. Let s_γ be as in the previous part. Then ${}^\sigma M(\lambda) = M(\lambda^{s_\gamma^{-1}})$.

From the Steinberg Tensor Product Theorem, it follows that any irreducible module M of ${}^tX_l(p^e)$ is stabilised by the field automorphism $\sigma = \phi^{et}$. Then by Proposition 4.1.25 and Corollary 1.7.20, it follows that M can be written over $\mathbb{F}_{p^{et}}$ (and potentially over a smaller field).

Many of the irreducible representations of ${}^tX_l(q)$ are tensor products (from the Steinberg Twisted Tensor Product Theorem), and thus although they are $\mathcal{S}_2$-candidates, they will be contained in $\mathcal{C}_4$ or $\mathcal{C}_7$. Following [8, Definition 5.1.15], we define a subclass $\mathcal{S}_2^*$ of $\mathcal{S}_2$ which consists of groups which are not obviously tensor products.

Proposition 4.1.26. *[8, Proposition 5.1.14] Let S be the image of an irreducible representation of ${}^t\hat{X}_l(q)$ over $\mathbb{F}_{q^t}$, with corresponding $\mathbb{F}_{q^t}$-module*

$$M = M_0 \otimes M_1^{\phi} \otimes \ldots \otimes M_{e-1}^{\phi^{e-1}}$$

as in Theorem 4.1.23. Suppose that S is an $\mathcal{S}_2$-candidate subgroup of a quasisimple classical group Ω, and let G be almost simple with socle $\bar{\Omega}$. If $N_G(\bar{S})$ is a maximal subgroup of G, then one of the following holds:

(i) Precisely one of the M_i is non-trivial.

(ii) Ω is defined over a proper subfield of $\mathbb{F}_{q^t}$.

(iii) Ω preserves an invariant classical form other than the induced form arising from the tensor factors M_i (if any).

(iv) (i), (ii) and (iii) do not occur and some outer automorphism of S that is induced by G does not permute the M_i. Note that this can only arise if ${}^tX_l(q) = \mathfrak{B}_2(2^e), \mathfrak{G}_2(3^e)$ or $\mathfrak{F}_4(2^e)$.

Definition 4.1.27. Let S, G and Ω be as in Proposition 4.1.26. Then $N_G(\bar{S})$ is an $\mathcal{S}_2^*$*-candidate subgroup of* G if (at least) one of the cases in Proposition 4.1.26 holds. The group $N_G(\bar{S})$ is an $\mathcal{S}_2^*$*-maximal subgroup of* G (or simply $\mathcal{S}_2^*$*-maximal*) if it is maximal amongst $\mathcal{S}_2^*$-candidate subgroups.

Case (i) in Proposition 4.1.26 consists of representations that are quasi-equivalent to a p-restricted representation. Such representations are classified in full generality for $\mathrm{SL}_2(q)$ in [6], and in low dimensions in [41], which will be sufficient for our purposes. Understanding the p-restricted representations will also allow us to construct any representations which may occur in the remaining cases.

The following proposition gives the precise conditions under which case (ii) of Proposition 4.1.26 occurs.

Proposition 4.1.28. *[8, Theorem 5.1.13] Let $q = p^e$, and suppose that ${}^tX_l(q)$ is not ${}^2\mathfrak{B}_2(q)$, ${}^2\mathfrak{F}_4(q)$, ${}^2\mathfrak{G}_2(q)$ or ${}^3\mathfrak{D}_4(q)$. Let $M \cong M_0 \otimes M_1^{\phi} \otimes \cdots \otimes M_{e-1}^{\phi^{e-1}}$, with each M_i a p-restricted module for ${}^tX_l(q)$. Suppose $f|te$, so that $\mathbb{F}_{p^f} \subset \mathbb{F}_{p^{te}}$. Then M can be realised over $\mathbb{F}_{p^f}$ if and only if one of the following conditions holds:*

(i) $t = 1$, *and* $M_i \cong M_j$ *whenever* $i \equiv j \mod f$.

(ii) $t = 2$, $f|e$, $M_i \cong M_i^{\phi^e}$ *for all* i *and* $M_i \cong M_j$ *whenever* $i \equiv j \mod f$.

(iii) $t = 2$, $f \nmid e$, $M_i \not\cong M_i^{\phi^e}$ *for some* i, $M_i \cong M_j$ *whenever* $i \equiv j \mod f$ *and* $M_i \cong M_j^{\phi^e}$ *whenever* $i \equiv j \mod \frac{f}{2}$ *but* $i \not\equiv j \mod f$.

In the following sections, we will find all the 16- and 17-dimensional $\mathcal{S}_2^*$-candidate subgroups for each group ${}^tX_l(q)$ in turn. The process is similar to that as described in Section 3.2. To find the list of $\mathcal{S}_2^*$-candidate subgroups, we use the computations above and the tables in [41]. We will often need to construct the modules in question explicitly, which will also determine the field of definition. To determine the form preserved by the image of the representation, we will make use of Lemma 1.7.7 and Lemma 1.7.8, as well as the following result:

Proposition 4.1.29. *[8, Proposition 5.1.12] and [29, Section 31.6]. All irreducible $\overline{\mathbb{F}}_p$-modules of quasisimple extensions of $\mathrm{S}_n(q)$ and $\mathrm{O}_n^{\epsilon}(q)$ are self-dual.*

For the actions of the automorphisms we will make use of Proposition 4.1.25 and the construction of the modules to determine precisely which outer automorphisms of G stabilise the representation in question. We will compute the stabilisers of these representations, as well as provide the explicit constructions, in Section 4.3, and summarise these results in Section 4.2. We will consider the actions of the graph and field automorphisms in Section 4.4, and containments between $\mathcal{S}_2^*$-candidate subgroups in Section 4.5. A summary of all the $\mathcal{S}_2^*$-maximal subgroups can be found in Section 4.6.

4.2 Table of candidates

Theorem 4.2.1. *Let S be an $\mathcal{S}_2^*$-candidate subgroup of a classical group in dimensions 16 or 17. Then $G = S^\infty$ is contained in Table 4.1.*

Proof. This will be shown in Section 4.3. □

We provide a brief explanation of the columns in Table 4.1:

- 'Group' denotes the isomorphism type of the $\mathcal{S}_2^*$-candidate subgroup S.
- 'Module' gives a brief description of the corresponding module. Further details for this can be found in the associated theorems.
- 'Cond' gives restrictions on q for the $\mathcal{S}_2^*$-candidate subgroup to exist and have the specified structure.
- 'Dim' denotes the dimension of the module.
- 'Case' determines the form preserved by S.
- 'c' denotes the number of conjugacy classes of S within the conformal classical group in question.
- 'Stab' denotes the class stabiliser within the conformal classical group.
- 'Theorems' denotes the references for theorems in later sections where S is constructed.

Table 4.1: $\mathcal{S}_2^*$-candidates

Group	Module	Cond	Dim	Case	c	Stab	Theorems
$\mathrm{L}_2(q).2$	$S^{16}(V)$	$p \geq 17$	17	O$^\circ$	2	1	4.3.7
$\mathrm{SL}_2(q)$	$S^{15}(V)$	$p \geq 17$	16	S	1	$\langle\delta\rangle$	4.3.7
$\mathrm{L}_2(q^4).4$	$V \otimes V^\sigma \otimes V^{\sigma^2} \otimes V^{\sigma^3}$	$p \neq 2$	16	O$^+$	4	$\langle\delta\rangle$	4.3.8
$\mathrm{L}_2(q^4).4$	$V \otimes V^\sigma \otimes V^{\sigma^2} \otimes V^{\sigma^3}$	$p = 2$	16	O$^+$	2	1	4.3.8
$\mathrm{L}_2(q^2).2$	$V_4 \otimes V_4^\sigma$	$p \neq 2, 3$	16	O$^+$	4	$\langle\delta\rangle$	4.3.10
$\mathrm{SL}_4(q).2$	Sub of $V^{\otimes 3}$	$p = 3, e$ odd	16	L	2	1	4.3.14, 4.4.5
$\mathrm{SL}_4(q).2$	Sub of $V^{\otimes 3}$	$p = 3, e \equiv 2 \mod 4$	16	L	4	$\langle\delta^4\rangle$	4.3.14, 4.4.5
$\mathrm{SL}_4(q)$	Sub of $V^{\otimes 3}$	$p = 3, e \equiv 0 \mod 4$	16	L	4	$\langle\delta^4\rangle$	4.3.14, 4.4.5
$\mathrm{SU}_4(q).(q+1,4)$	Sub of $V^{\otimes 3}$	$p = 3$	16	U	$(q+1,16)$	1	4.3.15, 4.4.6
$\mathrm{L}_4(q^2).2$	$V \otimes V^\sigma$	$p = 2$	16	L	1	1	4.3.17, 4.4.7
$\mathrm{L}_4(q^2).(4 \times 2)$	$V \otimes V^\sigma$	$q = 3 \mod 4$	16	L	2	1	4.3.17, 4.4.7
$2^{\cdot}\mathrm{L}_4(q^2).(4 \times 2)$	$V \otimes V^\sigma$	$q = 5 \mod 8$	16	L	4	1	4.3.17, 4.4.7
$2^{\cdot}\mathrm{L}_4(q^2).2^2$	$V \otimes V^\sigma$	$q = 9 \mod 16$	16	L	4	$\langle\delta^4\rangle$	4.3.17, 4.4.7
$2^{\cdot}\mathrm{L}_4(q^2).2$	$V \otimes V^\sigma$	$q = 1 \mod 16$	16	L	4	$\langle\delta^4\rangle$	4.3.17, 4.4.7
$\mathrm{L}_4(q^2).2$	$V \otimes (V^*)^\sigma$	$p = 2$	16	U	1	1	4.3.18, 4.4.8
$\mathrm{L}_4(q^2).(4 \times 2)$	$V \otimes (V^*)^\sigma$	$q = 1 \mod 4$	16	U	2	1	4.3.18, 4.4.8
$2^{\cdot}\mathrm{L}_4(q^2).(4 \times 2)$	$V \otimes (V^*)^\sigma$	$q = 3 \mod 8$	16	U	4	1	4.3.18, 4.4.8
$2^{\cdot}\mathrm{L}_4(q^2).2^2$	$V \otimes (V^*)^\sigma$	$q = 7 \mod 16$	16	U	4	$\langle\delta^4\rangle$	4.3.18, 4.4.8
$2^{\cdot}\mathrm{L}_4(q^2).2$	$V \otimes (V^*)^\sigma$	$q = 15 \mod 16$	16	U	4	$\langle\delta^4\rangle$	4.3.18, 4.4.8
$\mathrm{Sp}_4(q)$	Sub of $V_4 \otimes V_5$	$p \neq 2, 5$	16	S	1	$\langle\delta\rangle$	4.3.21, 4.4.9
$\mathrm{Sp}_4(q)$	$V_4 \otimes V_4^\gamma$	$p = 2$	16	O$^+$	2	1	4.3.22
$\mathrm{S}_4(q^2).2$	$V \otimes V^\sigma$	all	16	O$^+$	$2(q-1,2)$	$\langle\delta\rangle$	4.3.23
$\Omega_{10}^+(q)$	Half-spin	q even	16	L	1	1	4.3.27
$2^{\cdot}\Omega_{10}^+(q).2$	Half-spin	$q \equiv 3 \mod 4$	16	L	2	1	4.3.27
$2^{\cdot}\Omega_{10}^+(q).4$	Half-spin	$q \equiv 5 \mod 8$	16	L	4	1	4.3.27
$2^{\cdot}\Omega_{10}^+(q).2$	Half-spin	$q \equiv 9 \mod 16$	16	L	4	$\langle\delta^4\rangle$	4.3.27
$2^{\cdot}\Omega_{10}^+(q)$	Half-spin	$q \equiv 1 \mod 16$	16	L	4	$\langle\delta^4\rangle$	4.3.27
$\mathrm{Sp}_8(q)$	Spin	q even	16	O$^+$	1	1	4.3.28
$2^{\cdot}\Omega_9^\circ(q)$	Spin	q odd	16	O$^+$	4	$\langle\delta\rangle$	4.3.28
$\Omega_{10}^-(q)$	Half-spin	q even	16	U	1	1	4.3.29
$2^{\cdot}\Omega_{10}^-(q).2$	Half-spin	$q \equiv 1 \mod 4$	16	U	2	1	4.3.29
$2^{\cdot}\Omega_{10}^-(q).4$	Half-spin	$q \equiv 3 \mod 8$	16	U	4	1	4.3.29
$2^{\cdot}\Omega_{10}^-(q).2$	Half-spin	$q \equiv 7 \mod 16$	16	U	4	$\langle\delta^4\rangle$	4.3.29
$2^{\cdot}\Omega_{10}^-(q)$	Half-spin	$q \equiv 15 \mod 16$	16	U	4	$\langle\delta^4\rangle$	4.3.29

4.3 Constructing the candidates

In this section we provide constructions for each of the groups listed in Table 4.1. Let G be an $\mathcal{S}_i$-candidate subgroup, with $G < \Omega$ for some classical group Ω. For the $\mathcal{S}_1$-candidates it was most convenient to divide these into cases based on the structure of Ω, whereas when considering the $\mathcal{S}_2^*$-candidates here it is more convenient to base the cases on the structure of G (recall that G is also a group of Lie type for $\mathcal{S}_2^*$-candidates). In the following subsections we will consider each possible isomorphism type for G in turn, in each case determining all possible candidate subgroups, as well as the structure of $N_\Omega(G)$, and proving that this is all such candidates.

4.3.1 The groups $\mathrm{SL}_n(q)$ and $\mathrm{SU}_n(q)$

The groups $\mathrm{SL}_2(q)$

Definition 4.3.1. Let $q = p^e$ and $G = \mathrm{SL}_2(q)$. Let V be the natural 2-dimensional G-module defined over $\mathbb{F}_q$, and define V_1 to be the trivial G-module, $V_2 = V$ and $V_i = S^{i-1}(V)$ for $i > 2$. Let $n \in \mathbb{N}_0$, and write $n = a_0 + a_1 p + \cdots + a_s p^s$ for $a_i \in \{0, \ldots, p-1\}$. Then we define $M(n) = V_{a_0+1} \otimes V_{a_1+1}^{\phi} \otimes \cdots \otimes V_{a_s+1}^{\phi^s}$.

Remark 4.3.2. Note that $\dim V_i = i$ for all i, and $M(n)$ is a module of dimension $(a_0+1)\ldots(a_s+1)$.

Theorem 4.3.3 (Brauer and Nesbitt, [6])**.** *Let $q = p^e$ be a power of a prime p. Then a complete list of pairwise non-isomorphic absolutely irreducible $\mathrm{SL}_2(q)$-modules in characteristic p is given by $M(n)$ for $0 \leq n \leq q-1$. Further, $M(n)$ has highest weight n.*

Proposition 4.3.4. *[8, Corollary 5.3.3] With $M(n), a_i, q, p$ and e as in Definition 4.3.1 and Theorem 4.3.3 with $s = e-1$, the absolutely irreducible $\mathrm{SL}_2(q)$ module $M(n)$ has minimal field of realisation $\mathbb{F}_{p^f}$ if and only if $f | e$ and f is minimal such that $a_i = a_j$ whenever $i \equiv j \mod f$.*

Theorem 4.3.5. *Let G be an $\mathcal{S}_2^*$-candidate subgroup of a classical group Ω in dimensions 16 or 17 over $\mathbb{F}_q$ with nonabelian composition factor $S = \mathrm{L}_2(q^d)$ for some $d \geq 1$. Let M be the restriction of the natural module of Ω to G. Let σ denote the q-power Frobenius automorphism on $\mathrm{L}_2(q^d)$ of order d. Then d, S and M must be one of the four possibilities below:*

(i) $d = 1$, $S = \mathrm{L}_2(q)$, $M = V_{16}$, with $p > 16$ and $\dim M = 16$.

(ii) $d = 1$, $S = \mathrm{L}_2(q)$, $M = V_{17}$, *with* $p \geq 17$ *and* $\dim M = 17$.

(iii) $d = 2$, $S = \mathrm{L}_2(q^2)$, $M = V_4 \otimes V_4^\sigma$, *with* $p > 4$ *and* $\dim M = 16$.

(iv) $d = 4$, $S = \mathrm{L}_2(q^4)$, $M = V_2 \otimes V_2^\sigma \otimes V_2^{\sigma^2} \otimes V_2^{\sigma^3}$ *for all primes* p, *with* $\dim M = 16$.

Proof. Theorem 4.3.3 classifies all $\mathrm{SL}_2(q)$-modules consisting of a single non-trivial tensor factor. From Remark 4.3.2 we have that the p-restricted modules are of the form V_k (with weight $k-1$) for $k \leq p$, and $\dim V_k = k$. Hence the module $V_{16} = S^{15}(V)$ is a p-restricted irreducible representation when $p \geq 16$, and $V_{17} = S^{16}(V)$ is a p-restricted irreducible representation when $p \geq 17$; further, these are the only such p-restricted representations, and thus the only representations with precisely one non-trivial tensor factor. Thus it remains to consider modules with more than one non-trivial tensor factor.

If we have more than one non-trivial tensor factor then we are in the situation of Proposition 4.1.26 cases (ii) or (iii) (note that case (iv) cannot occur). For Case (ii) to occur, Proposition 4.3.4 tells us that a 16-dimensional module with more than one tensor factor must be either the $\mathrm{SL}_2(q^4)$ module $V_2 \otimes V_2^\sigma \otimes V_2^{\sigma^2} \otimes V_2^{\sigma^3}$, where V_2 is the natural module and occurs in all characteristics, or the $\mathrm{SL}_2(q^2)$ module $V_4 \otimes V_4^\sigma$, which occurs in characteristics $p \geq 5$ (in characteristics 2 and 3, the symmetric cube is reducible).

Otherwise, we are in case (iii) and the $\mathrm{SL}_2(q)$-module we are interested in must preserve a form other than the induced symplectic or symmetric bilinear form. If the module preserves a σ-hermitian form in addition to the induced form, then applying Lemma 1.7.10(ii) tells us that the group can be written over a smaller field - such groups have already been covered in case (ii). Otherwise, the only additional form the group can preserve is in characteristic 2. It follows from Proposition 1.8.7 and Theorem 4.3.3 that all 16-dimensional modules in characteristic 2 are of the form $W = \bigotimes_{j=1}^{4} V^{\phi^{i_j}}$ and preserve a quadratic form of plus type; however it also follows from Proposition 1.8.7 that $V^{\phi^{i_1}} \otimes V^{\phi^{i_2}}$ and $V^{\phi^{i_3}} \otimes V^{\phi^{i_4}}$ preserve quadratic forms of plus type; hence the action group of W is contained in $\Omega_4^+(q) \otimes \Omega_4^+(q)$, and hence is not maximal. $\square$

We will now consider each of the candidates in Theorem 4.3.5 in turn.

The p-restricted modules of $\mathrm{SL}_2(q)$ are considered in full generality in [8].

Proposition 4.3.6. *[8, Proposition 5.3.6]*

(i) If n *is even and* $p \geq n > 2$, *then there is a single conjugacy class of self-normalising* $\mathcal{S}_2^*$*-candidate subgroups of* $\Omega = \mathrm{Sp}_n(q)$ *isomorphic to* $S = \mathrm{SL}_2(q)$.

This is stabilised by δ_Ω and ϕ_Ω, which induce δ_S and ϕ_S respectively.

(ii) If $n \equiv \pm 3 \mod 8$ and $p \geq n$, then there is a single conjugacy class of self-normalising $\mathcal{S}_2^*$-candidate subgroups of $\Omega = \Omega_n^\circ(q)$ isomorphic to $S = \mathrm{L}_2(q)$. This is stabilised by δ_Ω and ϕ_Ω, which induce δ_S and ϕ_S respectively.

(iii) If $n \equiv \pm 1 \mod 8$ and $p \geq n > 1$, then there are exactly two conjugacy classes of self-normalising $\mathcal{S}_2^*$-candidate subgroups of $\Omega = \Omega_n^\circ(q)$ isomorphic to $S = \mathrm{L}_2(q).2$. They are interchanged by δ_Ω and stabilised by ϕ_Ω, which induces the field automorphism ϕ_S of S.

For ease of reference we extract the specific results we need.

Corollary 4.3.7. *Let G be an $\mathcal{S}_2^*$-candidate subgroup with composition factor $\mathrm{L}_2(q)$, whose corresponding module has dimension 16 or 17 and has precisely one non-trivial tensor factor.*

(i) If $p \geq 16$ then there is a single conjugacy class of self-normalising $\mathcal{S}_2^*$-candidate subgroups of $\Omega = \mathrm{Sp}_{16}(q)$ isomorphic to $S = \mathrm{SL}_2(q)$. This is stabilised by δ_Ω and ϕ_Ω, which induce δ_S and ϕ_S respectively.

(ii) If $p \geq 17$ there are exactly two conjugacy classes of self-normalising $\mathcal{S}_2^*$-candidate subgroups of $\Omega = \Omega_{17}^\circ(q)$ isomorphic to $S = \mathrm{L}_2(q).2$, which are interchanged by δ_Ω and stabilised by ϕ_Ω, which induces the field automorphism ϕ_S of S.

These are the only such $\mathcal{S}_2^*$-candidate subgroups.

Proposition 4.3.8. *Let $\Omega = \Omega_{16}^+(q)$ with $q = p^e$, let $G = \mathrm{L}_2(q^4)$ be an $\mathcal{S}_2^*$-candidate subgroup of Ω and let $S = N_\Omega(G)$. Assume Conjecture 2.3.3 holds Then $S = G.4$, where the automorphism of order 4 is the field automorphism σ_G. We have four (if p is odd) or two (if $p = 2$) conjugacy classes of $\mathcal{S}_2^*$-candidate subgroups isomorphic to $G.4$. When q is odd the class stabiliser in $\mathrm{CGO}_{16}^+(q)$ is $\langle \delta_\Omega \rangle$, with δ_Ω inducing δ_G; when q is even, the class stabiliser is trivial.*

Proof. From Theorem 4.3.5, we are interested in the $\mathrm{SL}_2(q^4)$-module

$$M = V \otimes V^\sigma \otimes V^{\sigma^2} \otimes V^{\sigma^3},$$

where V is the natural module of $\mathrm{SL}_2(q^4)$. First note that the centre of $\mathrm{SL}_2(q^4)$ consists of the matrices $\pm I_2$, and since $(-I_2)^{\otimes 4} = I_{16}$, the image of M is $\mathrm{L}_2(q^4)$.

In its natural representation, the group $\mathrm{SL}_2(q^4)$ preserves the symplectic form $\mathrm{antidiag}(-1, 1)$, which is also preserved by the action of σ. The Kronecker product of four such matrices gives a symmetric matrix f by Proposition 1.8.7 which is antidiagonal with determinant 1, a square, so that $G < \Omega_{16}^{+}(q^4)$. Since the automorphism $\sigma = \phi^e$ fixes this representation, it follows that we can find a matrix c such that G^c is written over $\mathbb{F}_q$, and this group preserves the form $\hat{f} := c^{-1}fc^{-T}$. Since $\det \hat{f} = \det f = 1$ it follows that $\hat{f}$ is also a form of type O^+ so that $G^c < \Omega_{16}^{+}(q)$.

Let ν be a primitive element of $\mathbb{F}_{q^4}$ and $g_\nu = \mathrm{diag}(\nu, 1)$ induce the diagonal automorphism δ_G on the natural representation of $\mathrm{SL}_2(q^4)$, which has order 2 unless q is even. This stabilises the representation. The corresponding 16-dimensional matrix $h_\nu := g_\nu \otimes g_\nu^\sigma \otimes g_\nu^{\sigma^2} \otimes g_\nu^{\sigma^3}$ scales the induced form by $\nu^{1+q+q^2+q^3} = \nu^{\frac{q^4-1}{q-1}}$ which is a primitive element of $\mathbb{F}_q^*$. Since δ_G preserves the form, we can choose c such that c also rewrites g_ν over a smaller field. Then since $g_\nu f g_\nu^T = \nu f$, we have $g_\nu^c \hat{f} g_\nu^{cT} = \nu \hat{f}$. Hence a matrix which induces δ_G lies inside $\mathrm{CGO}_{16}^{+}(q) \setminus \mathrm{GO}_{16}^{+}(q)$, and hence is induced by (a conjugate of) either δ_Ω or $\delta_\Omega\gamma_\Omega$. Note that $\delta_\Omega\gamma_\Omega$ has order 4 since $(\delta_\Omega\gamma_\Omega)^2 = \delta_\Omega'^2$, and hence cannot induce δ_G by Remark 3.4.7, so that δ_G is induced by δ_Ω.

Assuming Conjecture 2.3.3, it follows from Lemma 2.5.1 that the field automorphism σ is realisable over Ω.

The number of conjugacy classes is given by $\frac{|C:\Omega|}{|\delta_\Omega|}$ where C is the conformal group of Ω, which is 4 when q is odd and 2 when q is even.

□

Remark 4.3.9. If there are primes p for which Conjecture 2.3.3 does not hold, then the conclusion of Proposition 4.3.8 for these primes would read: Then $S = G.2$, where the automorphism of order 2 is the field automorphism σ_G^2. We have two (if p is odd) or one (if $p = 2$) conjugacy classes of $\mathcal{S}_2^*$-candidate subgroups isomorphic to S. When q is odd the class stabiliser in $\mathrm{CGO}_{16}^{+}(q)$ is $\langle \delta_\Omega, \delta_\Omega' \rangle$, with δ_Ω inducing δ_G and δ_Ω' inducing σ_G; when q is even, the class stabiliser is $\langle \gamma_\Omega \rangle$, with γ_Ω inducing σ_G.

Proposition 4.3.10. *Let $\Omega = \Omega_{16}^{+}(q)$ with $q = p^e$, $p \geq 5$, let $G = \mathrm{L}_2(q^2)$ be an $\mathcal{S}_2^*$-candidate subgroup of Ω, and let $S = N_\Omega(G)$. Assume Conjecture 2.3.3 holds. Then $S = G.2$, where the 2 corresponds to the field automorphism σ_G of $\mathrm{L}_2(q^2)$ of order 2. We have four Ω-classes of $\mathcal{S}_2^*$-candidate subgroups isomorphic to S, with class stabiliser in $\mathrm{CGO}_{16}^{+}(q)$ given by $\langle \delta_\Omega \rangle$, with δ_Ω inducing δ_G.*

Proof. The $\mathrm{SL}_2(q^2)$-module $V_4 = S^3(V)$ is irreducible if $p \geq 5$, and its action group has isomorphism type $\mathrm{SL}_2(q^2)$ by Proposition 4.3.6. Let $M = V_4 \otimes V_4^\sigma$. The centre of the action group of V_4 consists of the matrices $\pm I_4$; hence the centre of the action group of M will be trivial and we have a representation of $\mathrm{L}_2(q^2)$.

From [8, Proposition 5.3.5], the action group of V_4 preserves the symplectic form $\hat{f} = \mathrm{antidiag}(6, -2, 2, -6)$ with determinant 144 and entries over $\mathbb{F}_p$; hence V_4^σ preserves the same symplectic form. By Proposition 1.8.7 the corresponding tensor field group preserves an orthogonal form with determinant 144^8, a square; hence we have $G < \Omega_{16}^+(q^2)$. Rewriting this matrix over a smaller field via conjugation by a matrix $c \in \mathrm{GL}_{16}(q^2)$, the corresponding rewritten tensor field group preserves the form $c^{-1}(\hat{f} \otimes \hat{f})c^{-T}$, with determinant $144^8 \det c^{-2}$. By Lemma 2.3.2, $\det c \in \mathbb{F}_q$, so the form preserved by the rewritten tensor field group also has determinant a square in $\mathbb{F}_q$, so that $G^c < \Omega_{16}^+(q)$.

For the action group of V_4 to preserve the standard symplectic form $f = \mathrm{antidiag}(-1, -1, 1, 1)$, we can conjugate the action group by $\mathrm{diag}(1, -3, 1, 1)$. In the natural representation of $\mathrm{SL}_2(q^2)$, the diagonal automorphism is induced by the matrix $\mathrm{diag}(\nu, 1)$, where ν is a primitive element of $\mathbb{F}_{q^2}$, and it follows from considering the usual basis of $S^3(V)$ that the matrix inducing the diagonal automorphism on V_4 is $g_\nu := \mathrm{diag}(\nu^3, \nu^2, \nu, 1)$. This commutes with $\mathrm{diag}(1, -3, 1, 1)$ so that this also induces the diagonal automorphism δ_G on V_4 preserving the standard symplectic form. The Kronecker product $g_\nu \otimes g_\nu^\sigma$ has first entry ν^{3q+3} and last entry 1 on the diagonal, and hence scales the preserved form $f \otimes f$ (which has entry 1 in the first and last entry of the antidiagonal) by $\nu^{3q+3} = \nu^{3(q+1)}$. Notice that $\theta := \nu^{q+1}$ is a primitive element of $\mathbb{F}_q^*$. Rewriting over a smaller field does not change how g_ν scales the form, so g_ν scales the rewritten form by θ^3. Since squares in $\mathbb{F}_q^*$ are even powers of θ, the element θ^3 is not a square and hence we cannot rescale g_ν to preserve f. Hence the diagonal automorphism δ_G lies in $\mathrm{CGO}_{16}^+(q) \setminus \mathrm{GO}_{16}^+(q)$, and as in Proposition 4.3.8 this is induced by δ_Ω.

Assuming Conjecture 2.3.3, it follows from Lemma 2.5.1 that the field automorphism σ is realisable over Ω. If $q = p^e$, then it follows from Theorem 4.1.23 that ϕ^i does not stabilise the representation for $1 \leq i \leq e - 1$. $\square$

Remark 4.3.11. If there are primes p for which Conjecture 2.3.3 does not hold, then the conclusion of Proposition 4.3.10 for these primes would read: Then $S = G$. We have two conjugacy classes of $\mathcal{S}_2^*$-candidate subgroups isomorphic to S, with class stabiliser in $\mathrm{CGO}_{16}^+(q)$ given by $\langle \delta_\Omega, \delta'_\Omega \rangle$, with δ_Ω inducing δ_G and δ'_Ω inducing σ_G.

Remark 4.3.12. Let G denote the action group of $V \otimes V^\sigma \otimes V^{\sigma^2} \otimes V^{\sigma^3}$, rewritten over a smaller field by conjugation by a matrix c. Let ρ denote the corresponding representation, so that $g\rho = c^{-1}(g \otimes g^{\sigma} \otimes g^{\sigma^2} \otimes g^{\sigma^3})c$. Then we have that:

- $g({}^{\phi_G}\rho) = c^{-1}(g^{\phi} \otimes g^{\phi\sigma} \otimes g^{\phi\sigma^2} \otimes g^{\phi\sigma^3})c$.
- $g(\rho^{\phi_\Omega}) = c^{-\phi}(g^{\phi} \otimes g^{\phi\sigma} \otimes g^{\phi\sigma^2} \otimes g^{\phi\sigma^3})c^{\phi}$.

Thus it follows that the ϕ_G automorphism of $G\rho$ is induced by ϕ_Ω followed by conjugation by the matrix $c^{-\phi}c$. We do not currently have proofs of any results about the structure of $c^{-\phi}c$ in general, although computations suggest that in the situations of both Proposition 4.3.10 above and Proposition 4.3.23, ϕ_Ω induces ϕ_G.

Irreducible $\mathrm{SL}_n(q)$ and $\mathrm{SU}_n(q)$ modules with one tensor factor, $n \geq 3$

Theorem 4.3.13. *The only irreducible p-restricted $\mathrm{SL}_n^{\pm}(q)$-modules with $n \geq 3$ are the 16-dimensional representations of $\mathrm{SL}_4^{\pm}(q)$ with $q = 3^e$, with highest weight $(0,1,1)$.*

Proof. Direct from [41]. □

Proposition 4.3.14. *Let $\Omega = \mathrm{SL}_{16}(q)$ with $q = 3^e$, let $G = \mathrm{SL}_4(q)$ be an $\mathcal{S}_2^*$-candidate subgroup of Ω, and let $S = N_\Omega(G)$. Then we have:*

(i) If e is odd, then $S = G.2$ with the 2 automorphism induced by δ_G. We have two Ω-classes of subgroups isomorphic to S, with trivial class stabiliser in $\mathrm{GL}_{16}(q)$.

(ii) If $e \equiv 2 \mod 4$, then $S = G.2$ with the 2 automorphism induced by δ_G^2. We have four Ω-classes of subgroups isomorphic to S, with class stabiliser in $\mathrm{GL}_{16}(q)$ given by $\langle \delta_\Omega^4 \rangle$, with δ_Ω^4 inducing the δ_G automorphism on S.

(iii) If $e \equiv 0 \mod 4$, then $S = G$. We have four Ω-classes of subgroups isomorphic to S, with class stabiliser in $\mathrm{GL}_{16}(q)$ given by $\langle \delta_\Omega^4 \rangle$, with δ_Ω^4 inducing the δ_G automorphism on S.

Proof. Choose a basis e_0, e_1, e_2, e_3 of the natural module of $\mathrm{SL}_4(q)$. Let

$$q_{i,j,k} = \sum_{\sigma \in \mathrm{Sym}(i,j,k)} e_{i^\sigma} \otimes e_{j^\sigma} \otimes e_{k^\sigma}$$

and $Q = \{q_{i,j,k} | i,j,k \in \{0,1,2,3\}\}$. This gives a basis for a 16-dimensional vector space spanned by Q; 12 vectors from choosing two of i,j,k to be the same, and 4 from choosing them all to be distinct. Note that $q_{i,j,k}$ is unchanged under reordering of subscripts, and that $q_{i,i,i}$ is trivial.

This gives an $\mathrm{SL}_4(q)$ module; take $g \in \mathrm{SL}_4(q)$ and write $g = (g_{i,j})$, so $e_i^g = \sum_r g_{i,r} e_r$, and given some $q_{i,j,k} \in Q$, the action of g on $q_{i,j,k}$ is given by

$$\begin{aligned}
&\sum_{\sigma \in \mathrm{Sym}(i,j,k)} e_{i^\sigma}^g \otimes e_{j^\sigma}^g \otimes e_{k^\sigma}^g \\
&\qquad = \sum_{\sigma \in \mathrm{Sym}(i,j,k)} \left(\sum_{r=0}^{3} g_{i^\sigma,r} e_r \right) \otimes \left(\sum_{s=0}^{3} g_{j^\sigma,s} e_s \right) \otimes \left(\sum_{t=0}^{3} g_{k^\sigma,t} e_t \right) \\
&\qquad = \sum_{\sigma \in \mathrm{Sym}(i,j,k)} \sum_{r=0}^{3} \sum_{s=0}^{3} \sum_{t=0}^{3} g_{i^\sigma,r} g_{j^\sigma,s} g_{k^\sigma,t} e_r \otimes e_s \otimes e_t.
\end{aligned}$$

The coefficient of $e_r \otimes e_s \otimes e_t$ is $\sum_\sigma g_{i^\sigma,r} g_{j^\sigma,s} g_{k^\sigma,t}$, which is independent of the order that r, s, t appear in the index; in other words, the coefficients of $e_r \otimes e_s \otimes e_t$ and $e_{r^\theta} \otimes e_{s^\theta} \otimes e_{t^\theta}$ will be the same for every $\theta \in \mathrm{Sym}(r, s, t)$. Denote this coefficient by $\lambda_{r,s,t}$; then we have

$$\sum_\sigma e_{i^\sigma}^g \otimes e_{j^\sigma}^g \otimes e_{k^\sigma}^g = \sum_{r=0}^{3} \sum_{s=0}^{3} \sum_{t=0}^{3} \lambda_{r,s,t} \sum_{\theta \in \mathrm{Sym}(r,s,t)} e_{r^\theta} \otimes e_{s^\theta} \otimes e_{t^\theta}$$

which is clearly in the span of Q. Hence we have a representation, which we will denote ρ.

In the natural representation of $\mathrm{SL}_4(q)$, generators of the maximal torus T are $t_1 := \mathrm{diag}(1, 1, \nu, \nu^{-1})$, $t_2 := \mathrm{diag}(1, \nu, \nu^{-1}, 1)$ and $t_3 := \mathrm{diag}(\nu, \nu^{-1}, 1, 1)$, with a Borel subgroup B consisting of the lower triangular matrices. Further, each t_i occurs from a coroot. The natural representation thus has highest weight $(0, 0, 1)$, agreeing with the notation in [41]. Using the lexicographical ordering, it follows that the image $B\rho$ will also consist of lower triangular matrices and thus stabilises the subspace $\langle (1, 0, \ldots, 0) \rangle$. By Lemma 4.1.17 it follows that the highest weight can be determined from the first entries of $t_i\rho$; i.e. their actions on the element $q_{0,0,1}$, which is determined by its action on $e_0 \otimes e_0 \otimes e_1$. It follows that $t_1\rho$ has first entry 1, and $t_2\rho$ and $t_3\rho$ have first entry ν, so that this representation has weight $(0, 1, 1)$ and hence from [41] contains an irreducible 16-dimensional constituent. Since the module is 16-dimensional, it follows that it must be irreducible.

The centre of $\mathrm{SL}_4(q)$ consists of scalar matrices θI_4 where $\theta^4 = 1$. Such scalars induce up to $\theta^3 I_{64} = \theta^{-1} I_{64}$ on $V^{\otimes 3}$, and hence acts as $\theta^{-1} I_{16}$ on $G\rho$; in particular ρ is faithful and the image of the representation is $\mathrm{SL}_4(q)$. This representation is not self-dual since it has highest weight $(0, 1, 1)$ (see [41, Section 6.3]) - in particular it preserves no bilinear form. It is also not stabilised by any nontrivial field automorphism of G; hence we only need to investigate the diagonal automorphism of G.

Given this construction, let $g_\nu = \mathrm{diag}(\nu, 1, 1, 1) \in \mathrm{GL}_4(3^e)$ induce the diagonal automorphism δ_G on the natural representation of G, where ν is a primitive element of $\mathbb{F}_{3^e}$. Then, considering the action of g_ν on the 16-dimensional representation, we see that g_ν acts as a scalar on each of the basis elements, scaling by ν^n if e_0 appears n times in the tensor product. This means that the determinant of the induced action of $g_\nu\rho$ on $G\rho < \mathrm{SL}_{16}(3^e)$ is $\nu^{12} = (\nu^3)^4$. We are interested in whether there exists a scalar μ such that $\det \mu g_\nu = 1$; i.e. whether ν^{12} is a sixteenth power. The order of ν^{12} in $\mathbb{F}_{3^e}^*$ is $\frac{3^e-1}{(3^e-1,12)} = \frac{3^e-1}{(3^e-1,4)}$; when e is odd, ν^{12} has odd order and hence is a sixteenth power (in fact a 2^k-power for any k), whereas when e is even ν^{12} has even order, and is a fourth power but not an eighth power. In particular, δ_G cannot be realised over $\mathrm{SL}_{16}(3^e)$ when e is even. When e is even, δ_G has order 4, so we can also consider in a similar computation whether δ_G^2 can be realised over Ω. The matrix δ_G^2 has determinant whose order is $\frac{3^e-1}{(3^e-1,8)}$. When $e \equiv 2 \mod 4$, $3^e - 1 \equiv 8 \mod 16$ so that δ_G^2 has odd order, hence is a sixteenth power and can be realised over $\mathrm{SL}_n(q)$. When $e \equiv 0 \mod 4$, δ_G^2 has even order, so that no non-trivial power of δ_G can be realised over $\mathrm{SL}_n(q)$.

Since $g_\nu\rho$ has determinant generated by fourth powers, it follows that δ_G is induced by δ_Ω^4 . □

Proposition 4.3.15. *Let $\Omega = \mathrm{SU}_{16}(q)$ with $q = 3^e$, let $G = \mathrm{SU}_4(q)$ be an $\mathcal{S}_2^*$-candidate subgroup of Ω, and let $S = N_\Omega(G)$. Then we have $S = G.(q+1, 4)$. We have $(q+1, 16)$ Ω-classes of subgroups isomorphic to S, with trivial class stabiliser in* $\mathrm{CGU}_{16}(q)$.

Proof. We construct the representation ρ in the same way as in Proposition 4.3.14. The corresponding module M preserves the induced unitary form I_{16}, and hence is stabilised by the automorphism $\gamma_\Omega\sigma_\Omega$.

From considering the weight of such a module we can see that it is not self dual, hence the duality automorphism does not normalise the group, and since $M, M^* \not\cong M^{\phi^i}$ for $1 \le i \le e-1$ by the Steinberg Tensor Product Theorem (Theorem 4.1.20), neither do any field automorphisms of G.

The diagonal automorphism of $G\rho$ is induced by conjugation by the image under ρ of $g_{\nu^{q-1}} = \mathrm{diag}(\nu^{q-1}, 1, 1, 1)$, which is also diagonal. The unitary form f preserved by the module is also diagonal, which takes value 1 on elements of the form $\sum_\sigma e_{i^\sigma} \otimes e_{i^\sigma} \otimes e_{j^\sigma}$ with $i \neq j$, and value 2 on elements $\sum_\sigma e_{i^\sigma} \otimes e_{j^\sigma} \otimes e_{k^\sigma}$ with i, j, k distinct. Thus for instance the entry corresponding to the basis element

$\sum_\sigma e_{1^\sigma} \otimes e_{1^\sigma} \otimes e_{2^\sigma}$ is 1 for f and the induced 16-dimensional matrices of $g_{\nu^{q-1}}$ and $g^*_{\nu^{q-1}}$. Since we know $g_{\nu^{q-1}}$ lies in the conformal group we have that $g_{\nu^{q-1}}$ preserves f.

Similarly to Proposition 4.3.14, the induced matrix $g_{\nu^{q-1}}\rho$ has determinant $\nu^{12(q-1)}$, and we are interested in whether it can be rescaled to preserve the form and have determinant 1. Note that the scalars which preserve the form are precisely those of the form $\nu^{t(q-1)}$ for any $0 \leq t < q+1$; thus we are interested in whether we can choose t such that $12 + 16t \equiv 0 \mod q+1$. When e is even, $q+1 \equiv 2 \mod 4$; hence we can equivalently consider t such that $6 + 8t \equiv 0 \mod \frac{q+1}{2}$. Since $\frac{q+1}{2}$ is odd, 8 generates the additive group $\mathbb{Z}/\frac{q+1}{2}\mathbb{Z}$, so such a t exists, and so we can rescale $g_{\nu^{q-1}}\rho$ to preserve the form and have determinant 1. Similarly, if e is odd then $3^e + 1 \equiv 4 \mod 8$, and we can find t such that $3 + 4t \equiv 0 \mod \frac{q+1}{4}$, so again we can rescale $g_{\nu^{q-1}}\rho$ to preserve the form and have determinant 1.

The diagonal automorphism δ_Ω of Ω permutes the classes. □

Irreducible $\mathrm{SL}_n(q)$ and $\mathrm{SU}_n(q)$ modules with more than one tensor factor, $n \geq 3$

We first provide a list of all possible candidates in this section, following the proof of [8, Lemma 5.4.19].

Theorem 4.3.16. *Let $G = \mathrm{SL}^{\pm}_n(q)$ with $n \geq 3$, and $\rho : G \to \Omega$ be a 16- or 17-dimensional representation whose image is an $\mathcal{S}^*_2$-candidate subgroup of some classical group Ω, and suppose the corresponding module M decomposes into a product of more than one non-trivial tensor factor. Then $G = \mathrm{SL}_4(q)$ and either $\Omega = \mathrm{SL}_{16}(\sqrt{q})$ with M quasi-equivalent to $V \otimes V^{\phi^{e/2}}$ or $\Omega = \mathrm{SU}_{16}(\sqrt{q})$ with M quasi-equivalent to $V \otimes (V^*)^{\phi^{e/2}}$.*

Proof. Since we are interested in modules with more than one non-trivial tensor factor, we can only have 16-dimensional representations occuring here. There are no 2-dimensional representations of $\mathrm{SL}^{\pm}_n(q)$ for $n \geq 3$, hence the only way this case can occur is as a tensor product of two 4-dimensional modules. The only possibilities in this case are the natural 4-dimensional representations of $\mathrm{SL}^{\pm}_4(q)$, and hence the modules we are interested in are of the form $M = M_1 \otimes M_2$ when $G = \mathrm{SL}^{\pm}_4(q)$, and $M_i = V^{\phi^j}$ or $(V^*)^{\phi^j}$ for some j, and V the natural module of $\mathrm{SL}^{\pm}_4(q)$. By replacing M by a quasi-equivalent module we may assume that $M_1 = V$.

If $G = \mathrm{SU}_4(q)$ then by Proposition 4.1.26 the corresponding module $M = V \otimes M_2$ would have to either be expressible over a smaller field or preserve a classical form other than the induced unitary form. If M were expressible over a smaller field, then from Proposition 4.1.28 since we have two tensor factors we must have that G can be written over $\mathrm{SU}_{16}(\sqrt{q})$, with underlying module $V \otimes V^{\phi^e} = V \otimes V^*$. Proposition 4.1.28 would then imply that $V \cong V^*$ which is not true since V is not self-dual. Similarly, if M preserved a form other than the induced form, this form would have to be bilinear. Hence by Lemma 1.7.10 (ii) the group would be expressible over a smaller field, and we have already seen that such groups do not give rise to $\mathcal{S}_2^*$-candidates. Also, it follows from Theorem 4.1.23 that the module $V \otimes V^*$ is reducible, hence cannot give rise to an $\mathcal{S}_2^*$-candidate subgroup before writing over a smaller field. Hence $G = \mathrm{SU}_4(q)$ does not give rise to an $\mathcal{S}_2^*$-candidate subgroup of any classical group.

If $G = \mathrm{SL}_4(q)$ then the same restrictions as in the previous paragraph apply to M. If M can be written over a smaller field we must have $V \cong M_2^{\phi^{e/2}}$ and thus $M = V \otimes V^{\phi^{e/2}}$; since this is not preserved by the action of duality or duality composed with the field automorphism σ, we have that this preserves no form and thus gives an $\mathcal{S}_2^*$-candidate subgroup of $\mathrm{SL}_{16}(\sqrt{q})$.

Suppose M preserves a form. It cannot preserve a bilinear form since none of the candidate modules for M are self-dual. Hence it must preserve a unitary form; in particular we must have $M^{*\sigma} \cong M$, so $M_2 \cong (V^*)^{\phi^{e/2}}$ and so $M = V \otimes (V^*)^{\phi^{e/2}}$, giving us an $\mathcal{S}_2^*$-candidate subgroup of $\mathrm{SU}_{16}(\sqrt{q})$. $\square$

Proposition 4.3.17. *Let $\Omega = \mathrm{SL}_{16}(q)$ with $q = p^e$, let $G = \mathrm{L}_4(q^2)$ be the composition factor of an $\mathcal{S}_2^*$-candidate subgroup of Ω, with G the action group of the rewritten tensor field representation $V \otimes V^\sigma$, and let $S = N_\Omega(G)$. Then we have:*

(i) If $p = 2$ then $S = G.2$, with the 2 automorphism induced by σ_G. We have a single Ω-class of subgroups isomorphic to S, with trivial class stabiliser in $\mathrm{GL}_{16}(q)$.

(ii) If $q \equiv 3 \mod 4$ then $S = G.(4 \times 2)$, with the group 4×2 induced by $\langle \sigma_G, \delta_G \rangle$. We have two Ω-classes of subgroups isomorphic to S, with trivial class stabiliser in $\mathrm{GL}_{16}(q)$.

(iii) If $q \equiv 5 \mod 8$ then $S = 2^{\cdot}G.(4 \times 2)$, with the group 4×2 induced by $\langle \sigma_G, \delta_G \rangle$. We have four Ω-classes of subgroups isomorphic to S, with trivial class stabiliser in $\mathrm{GL}_{16}(q)$.

(iv) If $q \equiv 9 \mod 16$ then $S := 2^{\cdot}G.2^2$, with the group 2^2 induced by $\langle \sigma_G, \delta_G^2 \rangle$. We have four Ω-classes of subgroups isomorphic to S, with class stabiliser in $\mathrm{GL}_{16}(q)$ given by $\langle \delta_\Omega^4 \rangle$, with δ_Ω^4 inducing the δ_G automorphism on S.

(v) If $q \equiv 1 \mod 16$ then $S := 2^{\cdot}G.2$, with the 2 automorphism induced by σ_G. We have four Ω-classes of subgroups isomorphic to S, with class stabiliser in $\mathrm{GL}_{16}(q)$ given by $\langle \delta_\Omega^4 \rangle$, with δ_Ω^4 inducing the δ_G automorphism on S.

In all cases the δ_Ω automorphism permutes the conjugacy classes of subgroups.

Proof. $\mathrm{SL}_4(q^2)$ preserves no nonzero form; hence neither does the action group of the tensor product of the modules. If $p = 2$ then the center of G is trivial; otherwise any scalar $\lambda I_4 \in G$ (so that $\lambda^4 = 1$) becomes $\lambda^{q+1} I_{16}$ in $\mathrm{SL}_{16}(q^2)$. When $q \equiv 3 \mod 4$, it follows that $\lambda^{q+1} = 1$ and so the image of the representation is $\mathrm{L}_4(q^2)$; when $q \equiv 1 \mod 4$, the matrix λI_4 induces $-I_{16}$ if λ can be chosen with order 4; since we are in $\mathbb{F}_{q^2}^*$ this is always possible when q is odd. Hence we get the representation has isomorphism type $2^{\cdot}\mathrm{L}_4(q^2)$ if $q \equiv 1 \mod 4$, and $\mathrm{L}_4(q^2)$ otherwise. In all cases, writing over a smaller field does not change the isomorphism type of the group.

The automorphism δ_G of G is trivial if q is even, and otherwise is induced by an element $g_\nu = \mathrm{diag}(\nu, 1, 1, 1)$, where ν denotes a primitive element of $\mathbb{F}_{q^2}^*$. The matrix g_ν induces up to an element in $\mathrm{GL}_{16}(q^2)$ with determinant $\nu^{4(q+1)}$. This can be rescaled by μI_{16} to have determinant 1 if and only if we can choose μ which gives $\mu^{16}\nu^{4(q+1)} = 1$, and in order to be able to still rewrite the corresponding matrix μg_ν over $\mathbb{F}_q^*$, we require $\mu \in \mathbb{F}_q$.

Since ν is a primitive element of $\mathbb{F}_{q^2}^*$, we have that ν^{q+1} is a primitive element of $\mathbb{F}_q^*$, which we denote κ; thus we are interested in when κ^4 is a sixteenth power in $\mathbb{F}_q^*$. The element κ has order $q-1$, and so κ^4 has order $\frac{q-1}{(q-1,4)}$. When $q \not\equiv 1 \mod 8$ it follows that κ^4 has odd order and hence is a sixteenth power; when $q \equiv 1 \mod 8$, κ^4 has even order and so is not a sixteenth power. Hence we can realise δ_G by conjugation by a matrix over $\mathbb{F}_q^*$ with determinant 1 when $q \not\equiv 1 \mod 8$, whereas when $q \equiv 1 \mod 8$ we cannot realise δ_G over Ω. Similarly the determinant of δ_G^2 is κ^8 and has odd order unless $q \equiv 1 \mod 16$; thus when $q \equiv 9 \mod 16$ we have that δ_G^2 is realisable over Ω, whereas when $q \equiv 1 \mod 16$ no nontrivial power of δ_G can be realised over Ω. Since δ_G has determinant a fourth power of a primitive element, δ_G is induced by δ_Ω^4.

The field automorphism σ_G interchanges the tensor factors, and the corresponding matrix has determinant 1 by Corollary 2.2.2; hence σ_G is also realisable over Ω and commutes with δ_G.

The outer automorphism group of Ω contained in the conformal group of Ω is generated by δ_Ω, which has order $(q-1, 16)$. The number of conjugacy classes follows from the order of the class stabiliser, and δ_Ω permutes the classes where there is more than one. $\square$

Proposition 4.3.18. *Let $\Omega = \mathrm{SU}_{16}(q)$ with $q = p^e$, let $G = \mathrm{L}_4(q^2)$ be the composition factor of an $\mathcal{S}_2^*$-candidate subgroup of Ω, with G the action group of the rewritten tensor field representation $V \otimes (V^*)^\sigma$, and let $S = N_\Omega(G)$. Then we have:*

(i) If $p = 2$ then $S = G.2$, with the 2 automorphism induced by $\sigma_G\gamma_G$. We have a single Ω-class of subgroups isomorphic to S, with trivial class stabiliser in $\mathrm{CGU}_{16}(q)$.

(ii) If $q \equiv 1 \mod 4$ then $S = G.(4 \times 2)$, with the group 4×2 generated by $\langle \sigma_G\gamma_G, \delta_G \rangle$. We have two Ω-classes of subgroups isomorphic to S, with trivial class stabiliser in $\mathrm{CGU}_{16}(q)$.

(iii) If $q \equiv 3 \mod 8$ then $S = 2^{\cdot}G.(4 \times 2)$, with the group 4×2 generated by $\langle \sigma_G\gamma_G, \delta_G \rangle$. We have four Ω-classes of subgroups isomorphic to S, with trivial class stabiliser in $\mathrm{CGU}_{16}(q)$.

(iv) If $q \equiv 7 \mod 16$ then $S := 2^{\cdot}G.2^2$, with the group 2^2 generated by $\langle \sigma_G\gamma_G, \delta_G^2 \rangle$. We have four Ω-classes of subgroups isomorphic to S, with class stabiliser in $\mathrm{CGU}_{16}(q)$ *given by $\langle \delta_\Omega^4 \rangle$, with δ_Ω^4 inducing the δ_G automorphism on S.*

(v) If $q \equiv 15 \mod 16$ then $S := 2^{\cdot}G.2$, with the 2 automorphism induced by $\sigma_G\gamma_G$. We have four Ω-classes of subgroups isomorphic to S, with class stabiliser in $\mathrm{CGU}_{16}(q)$ *given by $\langle \delta_\Omega^4 \rangle$, with δ_Ω^4 inducing the δ_G automorphism on S.*

In all cases the δ_Ω automorphism permutes the conjugacy classes of subgroups.

Proof. The proof is similar to the previous proposition so we only sketch the details.

The representation $V \otimes (V^*)^\sigma$ preserves a unitary form since it is fixed under the action of duality followed by σ. The image of a scalar λI_4 is $\lambda^{1-q} I_{16}$ in $\mathrm{SU}_{16}(q)$, so the image of the representation is $2^{\cdot}\mathrm{L}_4(q^2)$ if $q \equiv 3 \mod 4$ and $\mathrm{L}_4(q^2)$ otherwise.

Let g_ν be as in Proposition 4.3.17, for q odd and ν a primitive element of $\mathbb{F}_q^*$. The image of $g_\nu\rho$ in $\mathrm{SL}_{16}(q^2)$ is diagonal, hence $\delta_G^{*\sigma} = \delta_G^{-q}$. The form preserved by the image of G is a permutation matrix which fixes the first and last entries; since the last entry in $g_\nu\rho$ is 1, we have that the last entry in the form is left unchanged; hence $g_\nu\rho$ preserves the form. Similarly to before, when q is odd we can see that $g_\nu\rho$ has determinant $\nu^{4(q-1)}$ and we can rescale $g_\nu\rho$ to be realisable over $\mathrm{SU}_{16}(q^2)$ unless $q \equiv 7 \mod 8$, and we can similarly realise $g_\nu^2\rho$ unless $q \equiv 15 \mod 16$. When $p = 2$ there are no non-trivial diagonal automorphisms.

The automorphism $\sigma\gamma_G$ interchanges the factors in the same way as σ did in the previous proposition, hence $\sigma\gamma_G$ is realisable over $\mathrm{SU}_{16}(q^2)$ (as a product of permutation matrices the unitary form will be preserved), and δ_G and $\sigma\gamma_G$ commute. □

4.3.2 The groups $\mathrm{Sp}_n(q)$ and $\mathrm{Sz}(2^{2e+1})$

Theorem 4.3.19. *Let $q = p^e$. Then the only $\mathcal{S}_2^*$-candidate subgroups of the classical groups in dimensions 16 and 17 with composition factor $\mathrm{S}_n(q)$ (for $n \geq 4$) are the following, all of which occur in dimension 16:*

(i) $G = \mathrm{Sp}_8(q)$ *with* $q = 2^e$.

(ii) $G = \mathrm{Sp}_4(q)$ *with* $p \neq 5$.

(iii) $G = \mathrm{Sp}_4(q^2)$, $M = V \otimes V^\sigma$ *rewritten over* $\mathbb{F}_q$, *where* V *is the natural module of* G.

Proof. From [41], the 16- and 17-dimensional p-restricted modules for $\mathrm{Sp}_n(q)$ when $n \geq 6$ are a 16-dimensional $\mathrm{Sp}_8(q)$-module for q even, and the natural 16-dimensional representation of $\mathrm{Sp}_{16}(q)$ (which we do not need to consider). The only 2-dimensional representation of a group $\mathrm{Sp}_n(q)$ is the natural 2-dimensional representation of $\mathrm{Sp}_2(q) \cong \mathrm{SL}_2(q)$ which we have already considered in Section 4.3.1. Thus it remains to consider all possible 16-dimensional modules of $\mathrm{Sp}_4(q)$.

For $\mathrm{Sp}_4(q)$, there is a 16-dimensional p-restricted representation in characteristics other than 5 (considered in Proposition 4.3.21).

The list of p-restricted 4-dimensional modules consists of the natural 4-dimensional module, which we denote V and which occurs in all characteristics; and when $p = 2$ the quasi-equivalent module V^γ, the image of the natural representation under the exceptional graph automorphism. By Theorem 4.1.23 (and replacing the module by a quasi-equivalent module), we may assume that a non-p-restricted 16-dimensional irreducible module is of the form $W = V \otimes M^{\phi^f}$ for some integer $f < e$, where $M \in \{V, V^\gamma\}$ when $p = 2$ and $M = V$ otherwise. We use Proposition 4.1.26 and consider each of the possible cases in turn. Case (i) of Proposition 4.1.26, considering the p-restricted representations, is described above.

For case (ii) to apply, we can apply Proposition 4.1.28 to conclude that the base field is $\mathbb{F}_{q^2}$ for some prime power q, and the module is $V \otimes V^\sigma$ where σ is the q-power Frobenius automorphism.

Note that W is self-dual, so for case (iii) to occur the module would also need to preserve a unitary form; then, by Lemma 1.7.10(ii) we can rewrite the module over a smaller field and hence we are in the situation of case (ii). Hence when p is odd these are the only possibilities.

In the case when $p = 2$, we also need to rule out case (iv). Note that the module we are considering in this case is $V \otimes V^{\gamma^f}$ for some f. If γ preserved the module and did not permute the tensor factors, then neither would γ^f, which is impossible. Hence case (iv) also cannot occur. $\square$

Remark 4.3.20. When q is even, $\mathrm{Sp}_8(q) \cong \Omega_9^\circ(q)$, and the 16-dimensional representation described in Theorem 4.3.19 is the spin representation. Thus we will consider this group in the section on spin representations.

Proposition 4.3.21. *Let $\Omega = \mathrm{Sp}_{16}(q)$ with $q = p^e$ and $p \neq 2, 5$, let $G = \mathrm{Sp}_4(q)$ be an $\mathcal{S}_2^*$-candidate subgroup of Ω, and let $S = N_\Omega(G)$. Then we have $S = G$. We have a single Ω-class of subgroups isomorphic to S, with class stabiliser in $\mathrm{CSp}_{16}(q)$ given by $\langle \delta_\Omega \rangle$, with δ_Ω inducing δ_G.*

Proof. Note that $\mathrm{S}_4(q) \cong \Omega_5^\circ(q)$. Let V_4 be the natural module of $\mathrm{Sp}_4(q)$ and V_5 the natural module of $\Omega_5(q)$, and define $W = V_4 \otimes V_5$. Since V_4 has weight $(0, 1)$ and when $p \neq 2$ V_5 has weight $(1, 0)$, it follows from Lemma 4.1.21 that $V_4 \otimes V_5$ has a constituent V_{16} with highest weight $(1, 1)$, which from [41] is 16-dimensional. Note that all irreducibles modules of $\mathrm{Sp}_4(q)$ are self-dual by Proposition 4.1.29; in addition, tensor products of such modules are self-dual, so that $V_4 \otimes V_5$ is self-dual. If V_{16} occurred as a quotient of $V_4 \otimes V_5$, then since duality is a contravariant exact functor (see for instance [42, Chapter I.8]) it follows that $V_{16}^* = V_{16}$ is a submodule of $(V_4 \otimes V_5)^* = V_4 \otimes V_5$; hence we have that V_{16} is a submodule of $V_4 \otimes V_5$. Similarly, if V_{16} were a quotient of a submodule of $V_4 \otimes V_5$ (which would also be self-dual as a submodule of a self-dual module), it would follow that V_{16} were also a submodule. Hence V_{16} is a submodule of $V_4 \otimes V_5$.

Next, we determine the action of the diagonal automorphism of $\mathrm{Sp}_4(q)$ on the representation. Let V_4 have basis v_0, v_1, v_2, v_3. We can realise V_5 as a submodule of $V_4 \otimes V_4$, with basis elements:

$$\begin{aligned} v_0 \otimes v_1 - v_1 \otimes v_0, \\ v_0 \otimes v_2 - v_2 \otimes v_0, \\ v_1 \otimes v_3 - v_3 \otimes v_1, \\ v_2 \otimes v_3 - v_3 \otimes v_2, \\ v_0 \otimes v_3 - v_3 \otimes v_0 + v_2 \otimes v_1 - v_1 \otimes v_2. \end{aligned}$$

To see this, note that the exterior square $E := \Lambda^2(V_4)$ is 6-dimensional and has a basis given by vectors $v_i \otimes v_j - v_j \otimes v_i$. The roots of the natural representation of $\mathrm{Sp}_4(q)$ are $\mathrm{diag}(\nu, \nu^{-1}, \nu, \nu^{-1})$ and $\mathrm{diag}(1, \nu, \nu^{-1}, 1)$, and since E is lower-triangular-preserving it follows from Lemma 4.1.17 that this representation has highest weight

$(0, 1)$. Hence from [41] it follows that E must have a 5-dimensional constituent V_5; since E is self-dual it follows that V_5 must exist as a submodule of E, and it is straightforward to check that the generators given above form such a submodule.

The diagonal automorphism of V_4 is induced by the matrix $g_\nu = \mathrm{diag}(\nu, \nu, 1, 1)$ for ν a primitive element of $\mathbb{F}_q^*$. The matrix g_ν scales the induced form by ν, and its action on the corresponding module V_5 is via the matrix $\mathrm{diag}(\nu^2, \nu, \nu, \nu, 1)$. The form preserved by V_5 is $\mathrm{antidiag}(1, -1, -2, -1, 1)$, and so the diagonal automorphism scales this by a factor ν^2. Thus the action of the diagonal automorphism on $V_4 \otimes V_5$ (and hence also on all submodules of $V_4 \otimes V_5$) is to multiply the form by ν^3, a non-square. Hence this is induced by the diagonal automorphism δ_Ω of $\mathrm{Sp}_{16}(q)$. The number of conjugacy classes follows. □

Proposition 4.3.22. *Let $\Omega = \Omega_{16}^+(q)$ with $q = 2^e$, and let $G = \mathrm{Sp}_4(q)$ be an $\mathcal{S}_2^*$-candidate subgroup of Ω with highest weight $(1, 1)$. Then G is not $\mathcal{S}_2^*$-maximal.*

Proof. Consider V_5, the natural representation of $\Omega_5^\circ(q) \cong \mathrm{S}_4(q)$, with basis $v_1, \ldots, v_5$ and preserving the standard antidiagonal form. Then V_5 has a 1-dimensional submodule given by $\langle v_3 \rangle$, and quotienting out by this module gives us a 4-dimensional irreducible module V_4' which is not isomorphic to V_4. The tensor product $W = V_4 \otimes V_4'$ is a 16-dimensional module, and since the weights of these modules are $(1, 0)$ and $(0, 1)$ respectively, it follows from Lemma 4.1.21 that W has a constituent with highest weight $(1, 1)$. However from [41] both W and the module of highest weight $(1, 1)$ have dimension 16, so that W is irreducible. Hence W is a tensor product, and does not satisfy any of the conditions of Proposition 4.1.28; thus W is not $\mathcal{S}_2^*$-maximal.

□

Proposition 4.3.23. *Let $\Omega = \Omega_{16}^+(q)$ with $q = p^e$, let $G = \mathrm{S}_4(q^2)$ be an $\mathcal{S}_2^*$-candidate subgroup of Ω, with G the action group of the rewritten tensor field representation $V \otimes V^\sigma$, and let $S = N_\Omega(G)$. Assume Conjecture 2.3.3 holds. Then we have $S = G.2$, with the 2 automorphism induced by σ. There are four (when p is odd) or two (when $p = 2$) Ω-classes of subgroups isomorphic to S. When p is odd the class stabiliser in $\mathrm{CGO}_{16}^+(q)$ is given by $\langle \delta_\Omega \rangle$, with δ_Ω inducing δ_G; when q is even, the class stabiliser in $\mathrm{CGO}_{16}^+(q)$ is trivial.*

Proof. Suppose first that q is odd. Let V denote the natural module of $\mathrm{Sp}_4(q^2)$; then we obtain the tensor field representation $V \otimes V^\sigma$ with corresponding tensor field group H, and rewritten tensor field group $G = H^c$ as a matrix group over $\mathbb{F}_q$, for some $c \in \mathrm{GL}_{16}(q^2)$. This gives rise to an $\mathcal{S}_2^*$-candidate subgroup of some classical group. Let G preserve the form $f = \mathrm{antidiag}(-1, -1, 1, 1)$; then $f \otimes f$ is a symmetric matrix with determinant 1, so that H preserves an orthogonal plus-type form over $\mathbb{F}_q$ by Proposition 1.8.7. Since the module is stabilised by σ there exists a matrix $c \in \mathrm{GL}_{16}(q^2)$ such that $H^c < \mathrm{SL}_{16}(q)$ with H^c preserving the form $c^{-1}(f \otimes f)c^{-T}$. This is clearly a symmetric matrix, so that H^c preserves an orthogonal form, and the form has determinant $\det c^{-2}$. By Lemma 2.3.2 we have $\det(c) \in \mathbb{F}_q$, so that G preserves a form of orthogonal plus type.

The centre of $\mathrm{Sp}_4(q^2)$ consists of $\pm I_4$, which induces up to I_{16}, so that the image of the representation is $\mathrm{S}_4(q^2)$.

The diagonal automorphism of G is induced by the matrix $g_\nu = \mathrm{diag}(\nu, \nu, 1, 1)$, which scales the form f by a factor of ν. It follows that the matrix $g_\nu \otimes g_\nu^\sigma$ scales the form $f \otimes f$ by a factor of ν^{q+1}, which is a primitive element of $\mathbb{F}_q^*$. Then $c^{-1}(g_\nu \otimes g_\nu^\sigma)c$ scales the form $c^{-1}(f \otimes f)c^{-T}$ preserved by H by the same factor ν^{q+1}, so that the automorphism induced by the diagonal automorphism of G lies in $\mathrm{CGO}_{16}^+(q) \setminus \mathrm{GO}_{16}^+(q)$. As in Proposition 4.3.8 this is induced by δ_Ω or $\delta_\Omega \gamma_\Omega$, and since δ_G has order 2 and $\delta_\Omega \gamma_\Omega$ has order 4, δ_G is induced by δ_Ω.

Assuming Conjecture 2.3.3, it follows from Lemma 2.5.3 that the matrix inducing the σ automorphism preserves the form and has determinant and spinor norm 1.

If $p = 2$, then we construct the module in the same way, and it follows from Proposition 1.8.7(iii) that the tensor field group preserves a quadratic form of plus type. By Lemma 2.3.1 the rewritten tensor group will also preserve an orthogonal form of plus type, so we again have $G < \Omega$. We have

$$\mathrm{Out}(G) = \langle \gamma, \delta | \gamma^2 = \phi, \phi^{2e} = 1 \rangle,$$

where γ is the exceptional graph automorphism of the Dynkin diagram $\mathfrak{B}_2(q)$ which interchanges the two roots. In particular from [41] we see that V is the module with highest weight $(1, 0)$, whereas V^γ is the module with highest weight $(0, 1)$, so that γ does not stabilise the module $V \otimes V^\sigma$. As before we have that no field automorphism ϕ_G^d stabilises the representation for $d < e$, and by Lemma 2.5.2 we have that σ has quasideterminant 1 and hence is induced by a matrix in Ω. □

Remark 4.3.24. As in Remark 4.3.12, we do not currently have a proof of the action of the field automorphism on the representation of $\mathrm{S}_4(q^2)$ as described above. If the conjecture is not true, the result will read similarly to the result as described in Remark 4.3.9.

When $q = 2^{2r+1}$, the group of Lie type $\mathfrak{B}_2(q)$ has an exceptional graph automorphism, which gives rise to a family of twisted groups ${}^2\mathfrak{B}_2(q)$ known as the

Suzuki groups. These occur naturally as a subgroup of $\mathrm{Sp}_4(2^{2e+1})$. Recall Theorem 4.1.24 which describes the representation theory of such groups.

Theorem 4.3.25. *Let $G = \mathrm{Sz}(2^{2e+1})$. Then there are no $\mathcal{S}_2^*$-candidate subgroups of any classical group Ω in dimensions 16 or 17 with composition factor G.*

Proof. The 16-dimensional representations are those which have two non-trivial 4-dimensional tensor factors in the above theorem. In order to be an $\mathcal{S}_2^*$-candidate subgroup we are in the situation of Proposition 4.1.26. Note that the Suzuki groups are defined over a field of order an odd power of 2; in particular such a finite field has no field automorphisms of order 2, so case (ii) of the proposition cannot occur. Case (iv) also cannot occur since the only outer automorphisms of the Suzuki groups are field automorphisms which permute the tensor factors. Since the natural module is self-dual by Proposition 4.1.29, it follows that all irreducible modules of $\mathrm{Sz}(2^{2e+1})$ in characteristic 2 are self-dual; thus in order for such a module to preserve a form other than the induced form, it would also need to preserve a unitary form; then by Lemma 1.7.10 the module would be expressible over a subfield, which is impossible. $\square$

4.3.3 The groups $\Omega_n^\epsilon(q)$

Theorem 4.3.26. *The only $\mathcal{S}_2^*$-candidate subgroups of the classical groups in dimensions 16 and 17 with composition factor $\mathrm{O}_n^\epsilon(q)$ (for $n \geq 7$) are the following, all of which occur in dimension 16:*

(i) $G = (2, q-1)^{\cdot}\Omega_9^\circ(q)$, with the module being the spin representation of G.

(ii) $G = (2, q-1)^{\cdot}\Omega_{10}^\pm(q)$ with the module being a half-spin representation of G.

Proof. We will not have any irreducible 2- or 4-dimensional representations in this case, since $n \geq 7$ and any group with such a representation will be isomorphic to a group discussed in a previous section. Thus we need only consider those groups whose module has a single irreducible tensor factor. From [41] we see that the only p-restricted representations to consider (excluding the natural representations of the 16- and 17-dimensional orthogonal groups) are the 16-dimensional representation of $\Omega_9^\circ(q)$ with highest weight $(1, 0, 0, 0)$ and two 16-dimensional representations of $\Omega_{10}^\pm(q)$ with highest weights $(1, 0, 0, 0, 0)$ and $(0, 1, 0, 0, 0)$ respectively, which are interchanged by the graph automorphism γ of $\Omega_{10}^\pm(q)$. The former is the spin representation $\mathrm{Spin}_9^\circ(q)$ and the latter modules are the half-spin representations $\mathrm{HSpin}_{10}^\pm(q)$. $\square$

Hence the only candidates that we need to consider in this section are the spin representations, which we will consider in full generality in Chapter 5. For convenience of reference, we quote below the specific cases of Lemma 5.2.13, Lemma 5.3.10, Lemma 5.3.11 and Lemma 5.4.14 that we need.

Lemma 4.3.27. *Let $\Omega = \mathrm{SL}_{16}(q)$ with $q = p^e$, let $G = (2, q-1)^{\cdot}\Omega^+_{10}(q)$ be an $\mathcal{S}_2^*$-candidate subgroup of Ω, with G the action group of the half-spin representation, and let $S = N_\Omega(G)$. Then we have:*

(i) If q is even then $S = G$. We have a single Ω-class of subgroups isomorphic to S, with trivial class stabiliser in $\mathrm{GL}_{16}(q)$.

(ii) If $q \equiv 3 \mod 4$ then $S = G.2$ with the 2 automorphism induced by δ_G. We have two Ω-classes of subgroups isomorphic to S, with trivial class stabiliser in $\mathrm{GL}_{16}(q)$.

(iii) If $q \equiv 5 \mod 8$ then $S = G.4$ with the 4 automorphism induced by δ_G. We have four Ω-classes of subgroups isomorphic to S, with trivial class stabiliser in $\mathrm{GL}_{16}(q)$.

(iv) If $q \equiv 9 \mod 16$ then $S = G.2$, with the 2 automorphism induced by $\delta_G^2 = \delta'_G$. We have four Ω-classes of subgroups isomorphic to S, with class stabiliser in $\mathrm{GL}_{16}(q)$ given by $\langle \delta_\Omega^4 \rangle$, with δ_Ω^4 inducing δ_G, an automorphism of order 4.

(v) If $q \equiv 1 \mod 16$ then $S = G$. We have four Ω-classes of subgroups isomorphic to S, with class stabiliser in $\mathrm{GL}_{16}(q)$ given by $\langle \delta_\Omega^4 \rangle$, with δ_Ω^4 inducing δ_G, an automorphism of order 4.

In all cases, γ_Ω induces γ_G and ϕ_Ω induces ϕ_G.

Lemma 4.3.28. *Let $\Omega = \Omega^+_{16}(q)$ with $q = p^e$, let $G = (2, q-1)^{\cdot}\Omega^\circ_9(q)$ be an $\mathcal{S}_2^*$-candidate subgroup of Ω, with G the action group of the spin representation, and let $S = N_\Omega(G)$. Then $S = G$. When q is odd there are four Ω-classes of subgroups isomorphic to S, with class stabiliser in $\mathrm{CGO}^+_{16}(q)$ given by $\langle \delta_\Omega \rangle$ with δ_Ω inducing δ_G. When q is even, there is a single Ω-class of subgroups isomorphic to $S \cong \mathrm{Sp}_8(q)$, with trivial class stabliser in $\mathrm{CGO}^+_{16}(q)$. In both cases ϕ_Ω induces ϕ_G.*

Lemma 4.3.29. *Let $\Omega = \mathrm{SU}_{16}(q)$ with $q = p^e$, let $G = (2, q-1)^{\cdot}\Omega^-_{10}(q)$ be an $\mathcal{S}_2^*$-candidate subgroup of Ω, with G the action group of the half-spin representation, and let $S = N_\Omega(G)$. Then we have:*

(i) If q is even then $S = G$. We have a single Ω-class of subgroups isomorphic to S, with trivial class stabiliser in $\mathrm{CGU}_{16}(q)$.

(ii) If $q \equiv 1 \mod 4$ *then* $S = G.2$ *with the* 2 *automorphism induced by* δ_G*. We have two* Ω*-classes of subgroups isomorphic to* S*, with trivial class stabiliser in* $\mathrm{CGU}_{16}(q)$*.*

(iii) If $q \equiv 3 \mod 8$ *then* $S = G.4$ *with the* 4 *automorphism induced by* δ_G*. We have four* Ω*-classes of subgroups isomorphic to* S*, with trivial class stabiliser in* $\mathrm{CGU}_{16}(q)$*.*

(iv) If $q \equiv 7 \mod 16$ *then* $S = G.2$*, with the* 2 *automorphism induced by* $\delta_G^2 = \delta'_G$*. We have four* Ω*-classes of subgroups isomorphic to* S*, with class stabiliser in* $\mathrm{CGU}_{16}(q)$ *given by* $\langle \delta_\Omega^4 \rangle$*, with* δ_Ω^4 *inducing* δ_G*, an automorphism of order 4.*

(v) If $q \equiv 15 \mod 16$ *then* $S = G$*. We have four* Ω*-classes of subgroups isomorphic to* S*, with class stabiliser in* $\mathrm{CGU}_{16}(q)$ *given by* $\langle \delta_\Omega^4 \rangle$*, with* δ_Ω^4 *inducing* δ_G*, an automorphism of order 4.*

In all cases, ϕ_Ω *induces* ϕ_G*.*

4.4 Graph and field automorphisms

In this section, we will consider the actions of graph and field automorphisms on some of the $\mathcal{S}_2^*$-candidate subgroups. Some of these computations have been done already:

(i) The graph and field automorphisms of the p-restricted representations of $\mathrm{SL}_2(q)$ are considered in Corollary 4.3.7.

(ii) The graph and field automorphisms of the spin representations are considered in Lemma 4.3.27, Lemma 4.3.28 and Lemma 4.3.29.

We also do not have results on the action of the field automorphism on rewritten tensor product groups preserving an orthogonal form, so we will not include these here.

We start the section with some general results, before providing specific computations for the remaining $\mathcal{S}_2^*$-candidate subgroups.

The following will be useful throughout.

Lemma 4.4.1. *[8, Lemma 5.1.6] There is a natural embedding of* $\hat{X}_l(q)$ *into* $\hat{X}_l(q^s)$ *for any* $s \geq 1$*, and the restriction of any* p*-restricted module for* $\hat{X}_l(q^s)$ *to* $\hat{X}_l(q)$ *is a* p*-restricted module of the same weight.*

Proof. Note that both $\hat{X}_l(q)$ and $\hat{X}_l(q^s)$ arise from restrictions of $\hat{X}_l(\overline{\mathbb{F}}_p)$ to centralisers of the field automorphisms ϕ^e and ϕ^{es} respectively; in particular this gives us the natural embedding of $\hat{X}_l(q)$ into $\hat{X}_l(q^s)$ since $\phi^{es} \in \langle \phi^e \rangle$. This proves the first part. If we have a p-restricted module of $\hat{X}_l(\overline{\mathbb{F}}_p)$ with weight λ, then its restriction to $\hat{X}_l(q)$ and $\hat{X}_l(q^s)$ will be p-restricted modules of the respective groups, also with weight λ, hence the second part holds. $\square$

Corollary 4.4.2. *Let $G_e = \hat{X}_l(p^e)$ be a group of Lie type over the field $\mathbb{F}_{p^e}$, and suppose G_e has an absolutely irreducible representation whose image is contained in an untwisted classical group Ω_e over $\mathbb{F}_{p^e}$, for every $e \geq 1$. Then the automorphism ϕ_G of G_e is induced by the field automorphism ϕ_Ω of Ω_e.*

Proof. First note that Ω_e has a subgroup Ω_1 obtained by restricting the field of definition, and this in turn has a subgroup G_1. Note also that we can obtain a subgroup of G_e by restricting the field in the natural representation of Ω_e, and from Lemma 4.4.1 it follows that this subgroup is also G_1.

The field automorphism ϕ_G of G_e is induced by ϕ_Ω followed by conjugation by a matrix g over $\mathbb{F}_{p^e}$; denote this automorphism by c_g. Since G_1 is fixed by ϕ_G, it follows that $\phi_\Omega c_g$ centralises G_1.

We also have ϕ_Ω centralises Ω_1, and hence ϕ_Ω also centralises G_1. Thus since the representation of G_1 is absolutely irreducible, it follows that $g = \lambda I_n$ for some scalar $\lambda \in \mathbb{F}_{p^e}^*$, so that ϕ_G is induced by ϕ_Ω. $\square$

We first consider those groups which are tensor powers of the natural module of $\mathrm{SL}_n(q^d)$. For these, the following general results will be useful:

Lemma 4.4.3. *Let $G = \mathrm{SL}_n(q^d)$, let V be the natural G-module with basis $e_1, \ldots, e_n$, and let $\rho : G \to \Omega := \mathrm{SL}_{n^d}(q)$ be a representation with corresponding module $W := V \otimes V^\sigma \otimes \ldots \otimes V^{\sigma^{d-1}}$ and image $G\rho := \hat{G}$ which can be rewritten over $\mathbb{F}_q$ by conjugation by a matrix $c \in \mathrm{GL}_{n^d}(q^d)$. Then:*

(i) For $g \in G$, the matrix $M(g)$ of the action of g on the rewritten tensor field representation is given by $c^{-1}(g \otimes g^\sigma \otimes \ldots \otimes g^{\sigma^{d-1}})c$.

(ii) For $n > 2$ and q odd, the automorphism γ_G is induced by γ_Ω followed by conjugation by $c^T c$, and in terms of outer automorphisms of Ω is induced by γ_Ω if $n \equiv 0, 1 \mod 4$ or if $n \equiv 2 \mod 4$ and $d > 2$ and by $\gamma_\Omega \delta_\Omega$ otherwise.

(iii) For q odd, the automorphism ϕ_G is induced by ϕ_Ω followed by conjugation by $c^{-\phi}c$, and in terms of outer automorphisms of Ω is induced by $\phi_\Omega \delta_\Omega^{\frac{(p-1,n^d)}{(2,n)}}$ if $n \equiv 2 \mod 4$ and $d = 2$, and ϕ_Ω otherwise.

Proof.

(i) This is clear from the construction of the module.

(ii) We have that

$$M(g)^{\gamma_\Omega} = (c^{-1}(g \otimes g^{\sigma} \otimes \ldots \otimes g^{\sigma^{d-1}})c)^{-T} = c^{T}(g^{-T} \otimes g^{-\sigma T} \otimes \ldots \otimes g^{-\sigma^{d-1}T})c^{-T}$$

whereas

$$M(g^{\gamma_G}) = c^{-1}(g^{-T} \otimes g^{-\sigma T} \otimes \ldots \otimes g^{-\sigma^{d-1}T})c,$$

and so $M(g^{\gamma_G}) = c^{-1}c^{-T}M(g)^{\gamma_\Omega}c^{T}c$. Thus the automorphism γ_G of G is induced by the product of the automorphism γ_Ω and the automorphism induced by conjugation by $c^{T}c$.

To show that $c^{T}c \in \mathrm{GL}_{n^d}(q)$, we have that $c^{T}c$ induces a module isomorphism between ${}^{\gamma_G}W$ and W^{γ_Ω}, both of which are absolutely irreducible modules over $\mathbb{F}_q$; hence by Lemma 1.7.12 (i) there must exist $\lambda \in \mathbb{F}_{q^d}$ such that $\lambda c^{-1}c^{-T} \in \mathrm{GL}_{n^d}(q)$. We then have $\lambda c^{-1}c^{-T} = (\lambda c^{-1}c^{-T})^{\sigma} = \lambda^{q}c^{-\sigma}c^{-\sigma T}$. Rearranging, we get that $I_{n^d} = \lambda^{q-1}cc^{-\sigma}c^{-\sigma T}c^{T} = \lambda^{q-1}cc^{-\sigma}(cc^{-\sigma})^{T}$.

By Lemma 2.3.2 we can rescale c such that $cc^{-\sigma}$ is a permutation matrix; hence $(cc^{-\sigma})^{T} = (cc^{-\sigma})^{-1}$ and so $\lambda^{q-1} = 1$. Thus $\lambda \in \mathbb{F}_q$ and so $c^{-1}c^{-T} \in \mathrm{GL}_{n^d}(q)$; hence so is its inverse $c^{T}c$.

In order to decide which automorphism conjugation by $c^{T}c$ induces, we need to find $\det(c^{T}c)$. We have that $cc^{-\sigma}$ is the permutation matrix from Lemma 2.2.1, which we denote g_σ, and the conditions on n and d given in the statement determine whether $\det g_\sigma$ is 1 or -1. If $\det g_\sigma = 1$, this means that $\det(c)^{q-1} = 1$ and so $\det(c) \in \mathbb{F}_q$. In particular we have $\det c^{T}c = \det c^{2}$ is a square in $\mathbb{F}_q$. Hence by Lemma 3.2.14 γ_G is induced by a conjugate of γ_Ω. If $\det g_\sigma = -1$ then we have $\det c^{q-1} = -1$, so that $\det c \notin \mathbb{F}_q$ (since elements of $\mathbb{F}_q$ have multiplicative order dividing $q-1$) but $\det c^{2} \in \mathbb{F}_q$ (since $\det(c^{2})^{q-1} = 1$); in particular this means that $\det(c^{T}c)$ is not a square in $\mathbb{F}_q$, and so again by Lemma 3.2.14 we have that γ_G is induced by a conjugate of $\gamma_\Omega\delta_\Omega$.

(iii) Taking ϕ as the p-power automorphism, we have that

$$M(g)^{\phi} = (c^{-1}(g \otimes g^{\sigma} \otimes \ldots \otimes g^{\sigma^{d-1}})c)^{\phi} = c^{-\phi}(g^{\phi} \otimes g^{\phi\sigma} \otimes \ldots \otimes g^{\phi\sigma^{d-1}})c^{\phi}$$

whilst

$$M(g^\phi) = c^{-1}(g^\phi \otimes g^{\phi\sigma} \otimes \ldots \otimes g^{\phi\sigma^{d-1}})c,$$

and so $M(g^\phi) = c^{-1}c^\phi M(g)^\phi c^{-\phi}c$. Thus we have that the automorphism ϕ_G of G is induced by the automorphism ϕ_Ω, followed by conjugation by the matrix $c^{-\phi}c$.

To show that $c^{-\phi}c$ is written over $\mathbb{F}_q$, recall from the proof of part (ii) that $cc^{-\sigma} = g_\sigma$ is a permutation matrix. In particular $cc^{-\sigma} \in \mathrm{GL}_{n^d}(p)$, so that $cc^{-\sigma} = (cc^{-\sigma})^\phi = c^\phi c^{-\sigma\phi}$. Rearranging, we get that $c^{-\phi}c = c^{-\sigma\phi}c^{-\sigma} = (c^{-\phi}c)^\sigma$, so that $c^{-\phi}c \in \mathrm{GL}_{n^d}(q)$, and so as before $c^{-\phi}c$ induces some power of the diagonal automorphism. Note that here the conditions of Lemma 3.2.14 will not generally hold, so we must compute directly.

We have $\det(c^{-\phi}c) = \det(c)^{1-p}$. When $\det g_\sigma = -1$, we have that $\det c \notin \mathbb{F}_q$, but $\det c^2 \in \mathbb{F}_q$, so that $\det c^{p-1} \in \mathbb{F}_q$ is a $\frac{(p-1,n^d)}{(2,n)}$-power of a primitive element in $\mathbb{F}_q$, and hence ϕ_G is induced by $\phi_\Omega \delta_\Omega^{\frac{(p-1,n^d)}{(2,n)}}$. When $\det g_\sigma = 1$ it follows as before that $\det c \in \mathbb{F}_q$, and so when q is odd the determinant of $c^{-\phi}c$ is a $(p-1, n^d)$-power.

Recall that

$$\mathrm{Out}(\Omega) = \langle \delta_\Omega, \gamma_\Omega, \phi_\Omega | \delta_\Omega^{(q-1,n^d)} = \gamma_\Omega^2 = \phi_\Omega^e = [\gamma_\Omega, \phi_\Omega] = 1, \delta_\Omega^{\gamma_\Omega} = \delta_\Omega^{-1}, \delta_\Omega^{\phi_\Omega} = \delta_\Omega^p \rangle.$$

It follows that a typical element of $\mathrm{Out}(\Omega)$ can be written in the form $\gamma_\Omega^i \phi_\Omega^j \delta_\Omega^k$, and $\phi_\Omega^{\gamma_\Omega^i \phi_\Omega^j \delta_\Omega^k} = \phi_\Omega^{\delta_\Omega^k}$. The last relation in the automorphism group gives us that $\phi_\Omega^{-1}\delta_\Omega\phi_\Omega = \delta_\Omega^p$, so that $\delta_\Omega\phi_\Omega\delta_\Omega^{-1} = \phi_\Omega\delta_\Omega^{p-1}$ and so $\phi_\Omega^{\delta_\Omega^k} = \phi_\Omega\delta_\Omega^{k(1-p)}$. Thus the conjugates of ϕ_Ω are of the form $\phi_\Omega\delta_\Omega^{i(p-1)}$ and thus consist of the product of the automorphism ϕ_Ω by matrices whose determinants are a $(p-1,(q-1,n^d))$-power, and since $p-1|q-1$, this is the same as matrices whose determinants are a $(p-1,n^d)$-power. Thus from above, when $\det g_\sigma = 1$ we have that ϕ_G is induced by a conjugate of ϕ_Ω. We saw above that when $\det g_\sigma = -1$, ϕ_G is induced by $\phi_\Omega\delta_\Omega^{\frac{(p-1,n^d)}{(2,n)}}$, and it remains to see when this is conjugate to ϕ_Ω.

When $n \equiv 3 \mod 4$ then $\frac{(p-1,n^d)}{(2,n)} = (p-1, n^d)$ so in this case $\det c^{-\phi}c$ is a $(p-1,n^d)$-power, so that ϕ_G is induced by a conjugate of ϕ_Ω as well.

It remains to prove whether ϕ_Ω and $\phi_\Omega\delta_\Omega^{\frac{(p-1,n^d)}{(2,n)}}$ are conjugate when $n \equiv 2 \mod 4$ and $d = 2$, so we are considering these as elements of $\mathrm{Out}(\mathrm{SL}_{n^d}(q))$.

Similarly to before we have that $(\phi_\Omega \delta_\Omega^{\frac{(p-1,n^d)}{(2,n)}})^{\gamma_\Omega^i \phi_\Omega^j \delta_\Omega^k} = \phi_\Omega \delta_\Omega^{p^j(-1)^i \frac{(p-1,n^d)}{(2,n)} + (1-p)k}$, and this is conjugate to ϕ_Ω if and only if we can find $i, j, k, m \in \mathbb{Z}$ such that $p^j(-1)^i \frac{(p-1,n^d)}{(2,n)} = m(p^e - 1, n^2) - k(1-p)$. If $p \equiv 3 \mod 4$ then the left hand side is odd and the right hand side is even for every choice of i, j, k, m, whilst if $p \equiv 1 \mod 4$ then the right hand side is divisible by 4 whilst the left hand side is divisible by 2 but not 4. Hence ϕ_Ω and $\phi_\Omega \delta_\Omega^{\frac{(p-1,n^d)}{(2,n)}}$ are not conjugate.

□

The below result will be sufficient for our purposes when the representation of G preserves a unitary form.

Lemma 4.4.4. *Let $G = \mathrm{SL}_n(q^2)$, let V be the natural G-module with basis $e_1, \ldots, e_n$, and let $\rho : G \to \mathrm{SL}_{n^2}(q^2)$ be a representation with corresponding module $W := V \otimes (V^*)^\sigma$, where σ denotes the q-power field automorphism of G. Let $M(x)$ denote the matrix of the action of $x \in G$ on W. Then:*

(i) $M(x) = x \otimes x^{-\sigma T}$, and $M(x^*) = M(x)^*$.

(ii) $G\rho < \Omega := \mathrm{SU}_{n^2}(q, g_\sigma)$, where g_σ is the permutation matrix from Corollary 2.2.2.

(iii) The automorphism ϕ_G is induced by $\phi_\Omega \delta_\Omega$ if $n \equiv 2 \mod 4$ and $p \equiv 3 \mod 4$, and ϕ_Ω otherwise.

Proof.

(i) The first part is clear from the definition of W. For the second, note that $M(x)^* = (x \otimes x^{-\sigma T})^* = x^{-\sigma T} \otimes x = M(x^*)$.

(ii) Since $(W^*)^\sigma = W$, it follows from Lemma 1.7.8 that $G\rho$ must preserve a unitary form. We have

$$M(x)^{-\sigma T} = M(x)^* = x^{-\sigma T} \otimes x = g_\sigma^{-1}(x \otimes x^{-\sigma T})g_\sigma = g_\sigma^{-1} M(x) g_\sigma,$$

where g_σ is the permutation matrix from Lemma 2.2.1 which acts on W by permuting the tensor factors. Rearranging gives us that $g_\sigma = M(x) g_\sigma M(x)^{\sigma T}$, so that $M(x)$ preserves the unitary form g_σ.

(iii) In the notation of Lemma 3.2.16, we may set $\alpha = \phi_G$ and $\beta = \phi_\Omega$. Then since $M(x)^{\phi_\Omega} = x^{\phi_\Omega} \otimes x^{-\phi_\Omega \sigma T} = M(x^{\phi_G})$ and the form preserved is defined over

$\mathbb{F}_p$, we have that $L = I_{n^2}$ and $\lambda = 1$ (using the notation of Lemma 3.2.16). Hence, by Lemma 3.2.16 we have $\kappa = 1$ and the automorphism inducing ϕ_G depends on whether $\det g_\sigma^{\frac{1-p}{2}}$ is equal to 1 or -1. When $n \equiv 0, 1, 3 \mod 4$, $\det g_\sigma = 1$ and so in this case ϕ_G is induced by a conjugate of ϕ_Ω. When $n \equiv 2 \mod 4$, we have $\det g_\sigma = -1$ (since in the notation of Lemma 2.2.1, $d = 2$), so that when $p \equiv 1 \mod 4$ ϕ_G is induced by a conjugate of ϕ_Ω, and when $p \equiv 3 \mod 4$ ϕ_G is induced by a conjugate of $\phi_\Omega \delta_\Omega$.

□

Lemma 4.4.5. *Let G be an $\mathcal{S}_2^*$-candidate subgroup of $\Omega = \mathrm{SL}_{16}(3^e)$ with composition factor $\mathrm{L}_4(3^e)$ as in Proposition 4.3.14. Then the automorphisms γ_Ω and ϕ_Ω induce γ_G and ϕ_G respectively on G.*

Proof. The fact that ϕ_Ω induces ϕ_G follows from Corollary 4.4.2. We use a similar construction to consider the action of the graph automorphism.

Note that Ω has a subgroup $\hat{\Omega}$ isomorphic to $\mathrm{SL}_{16}(3)$ by restricting the field of definition, and this in turn has an $\mathcal{S}_2$-subgroup $\hat{G} = \mathrm{SL}_4(3)$. Note also that $\hat{G}$ is a subgroup of G obtained by restricting the field in the natural representation of $\mathrm{SL}_4(3^e)$. The duality automorphism γ_G of G is induced by γ_Ω followed by conjugation by some matrix g over $\mathbb{F}_{3^e}$. The action of $\gamma_\Omega c_g$ on G restricts to the action of the duality automorphism $\gamma_{\hat{G}}$ on $\hat{G}$, since duality on G restricts to duality on $\hat{G}$. Similarly, the automorphism $\gamma_{\hat{G}}$ of $\hat{G}$ is induced by $\gamma_{\hat{\Omega}} c_{\hat{g}}$ for some matrix $\hat{g}$ over $\mathbb{F}_3$. Thus $\gamma_\Omega c_g$ and $\gamma_{\hat{\Omega}} c_{\hat{g}}$ both induce $\gamma_{\hat{G}}$ on $\hat{G}$, and since γ_Ω induces $\gamma_{\hat{\Omega}}$ on $\hat{\Omega}$ it follows that g and $\hat{g}$ must be the same up to scalar multiplication. Since scalar multiplication does not affect the conjugation action of the matrix, we may assume that $g = \hat{g}$ and it suffices to consider the action in the case where $q = 3$. Computer computations in `sl4gamma` show that $\det \hat{g} = 1$ so that $\gamma_{\hat{\Omega}}$ induces $\gamma_{\hat{G}}$ on $\hat{G}$ and so γ_Ω induces γ_G on G.

□

Lemma 4.4.6. *Let G be an $\mathcal{S}_2^*$-candidate subgroup of $\Omega = \mathrm{SU}_{16}(3^e)$ with composition factor $\mathrm{U}_4(3^e)$ as in Proposition 4.3.15. Then the automorphism ϕ_Ω induces ϕ_G on G.*

Proof. Note that G and Ω occur naturally as subgroups of $\bar{G} = \mathrm{SL}_4(3^{2e})$ and $\bar{\Omega} = \mathrm{SL}_{16}(3^{2e})$ respectively; furthermore, since by Proposition 4.3.15 the form preserved by Ω and G is over the base field $\mathbb{F}_3$, the field automorphisms $\phi_{\bar{G}}$ and $\phi_{\bar{\Omega}}$ restrict to the field automorphisms ϕ_G and ϕ_Ω respectively. From Lemma 4.4.5 we have that $\phi_{\bar{\Omega}}$ induces $\phi_{\bar{G}}$. Thus ϕ_Ω induces ϕ_G on G as required.

□

Proposition 4.4.7. *Let G be an $\mathcal{S}_2^*$-candidate subgroup of $\Omega = \mathrm{SL}_{16}(q)$ with composition factor $\mathrm{L}_4(q^2)$ as in Proposition 4.3.17. Then the automorphisms γ_Ω and ϕ_Ω induce γ_G and ϕ_G respectively on G.*

Proof. This is direct from the table of candidates and Lemma 4.4.3. □

Proposition 4.4.8. *Let G be an $\mathcal{S}_2^*$-candidate subgroup of $\Omega = \mathrm{SU}_{16}(q)$ with composition factor $\mathrm{L}_4(q^2)$ as in Proposition 4.3.18. Then the automorphism ϕ_Ω induces ϕ_G on G.*

Proof. This is direct from the table of candidates and Lemma 4.4.4. □

Lemma 4.4.9. *Let G be an $\mathcal{S}_2^*$-candidate subgroup of $\Omega = \mathrm{Sp}_{16}(q)$ with composition factor $\mathrm{S}_4(q)$ as in Proposition 4.3.21. Then the automorphism ϕ_Ω induces ϕ_G on G.*

Proof. Direct from Corollary 4.4.2. □

4.5 Containments

In this section we check containments of $\mathcal{S}_2^*$-candidate subgroups in other $\mathcal{S}_2^*$-candidate subgroups. Note that we only have one $\mathcal{S}_2^*$-candidate subgroup in dimension 17, and no candidates as subgroups of $\Omega_{16}^-(q)$, so we have no containments to check in these cases.

We begin with some useful general results.

Lemma 4.5.1. *[48, Lemma 10.3.1] Let $G = Z(G).S$ be a quasisimple group, where S is a nonabelian simple group, and suppose $G < H$. Then for some $K < Z(G)$, $K.S$ embeds into a nonabelian composition factor of H.*

Lemma 4.5.2. *Let Ω be a classical group, and G and H groups with corresponding faithful representations ρ_G and ρ_H respectively, such that $G\rho_G$ and $H\rho_H$ are subgroups of Ω. Suppose we have a containment $H\rho_H \leq G\rho_G$, and that ρ_H is irreducible (respectively absolutely irreducible). Then for any group M with $H \leq M \leq G$, there exists an irreducible (respectively absolutely irreducible) representation ρ_M of M with $M\rho_M < \Omega$.*

Proof. $\rho_M = \rho_G|_M$, and clearly ρ_M is irreducible since $\rho_M|_H = \rho_H$, an irreducible representation. If ρ_H is absolutely irreducible, then ρ_M must also be absolutely irreducible; otherwise, by Lemma 1.7.3 there exist non-scalar matrices which centralise $M\rho_M$, and hence centralise $H\rho_H$, contradicting absolute irreducibility of ρ_H. □

Corollary 4.5.3. *Let G and H be quasisimple groups, with corresponding faithful absolutely irreducible representations ρ_G and ρ_H respectively such that $H\rho_H < G\rho_G < \Omega$ for some classical group Ω. Suppose additionally that there is a containment of H inside a subgroup C of G. Then C must only be centralised by scalars.*

Proof. By Lemma 4.5.2, there must exist an absolutely irreducible faithful representation ρ_C of C which embeds into Ω. Since ρ_C is absolutely irreducible it follows from Lemma 1.7.3 that $C\rho_C$ is centralised by scalars in Ω, and thus from the construction of ρ_C we must have that C is also centralised by scalars. □

Proposition 4.5.4. *Let $\Omega = \mathrm{Sp}_{16}(q)$. Then there are no containments between $\mathcal{S}_2^*$-candidate subgroups of Ω.*

Proof. From Theorem 4.2.1 we see that the $\mathcal{S}_2^*$-candidate subgroups are $\mathrm{SL}_2(q)$ for $p \geq 17$ and $\mathrm{Sp}_4(q)$ for $p \neq 2, 5$, so the only possibility for containment is when $p \geq 17$ and $\mathrm{SL}_2(q) < \mathrm{Sp}_4(q)$.

From [8, Section 8.2, Table 8.12] we see that $\mathrm{Sp}_4(q)$ has a number of subgroups isomorphic to $\mathrm{SL}_2(q)$. By Corollary 4.5.3 for there to be a containment the subgroup must only be centralised by scalars. Hence for instance there is no containment of $\mathrm{SL}_2(q)$ inside the $\mathcal{C}_2$-subgroup $\mathrm{Sp}_2(q)^2 : 2$ since any copy of $\mathrm{SL}_2(q)$ would be centralised by $\mathrm{Sp}_2(q)$. The other $\mathcal{C}_2$-subgroup of $\mathrm{Sp}_4(q)$, $\mathrm{GL}_2(q).2$, has an index-2 subgroup which is reducible in the natural representation of $\mathrm{Sp}_4(q)$ and hence is centralised by non-scalar elements; thus the 16-dimensional representation of $\mathrm{GL}_2(q)$ will also not be only centralised by scalars, ruling out a containment here also. This, along with Lagrange, rules out all possible containments except the $\mathcal{S}$-subgroup of $\mathrm{Sp}_4(q)$.

From Theorem 4.3.3 and Proposition 4.3.6, this subgroup is the action group of $S^3(W)$ where W is the natural module of $\mathrm{SL}_2(q)$, and $S^3(W)$ has highest weight 3. We obtain the 16-dimensional representation of $\mathrm{Sp}_4(q)$ by taking a submodule of the tensor product $V_4 \otimes V_5$ where V_4 is the natural module of $\mathrm{Sp}_4(q)$ and V_5 is the natural module of $\Omega_5^\circ(q)$, these groups being isomorphic. Again by Theorem 4.3.3 and Proposition 4.3.6, $\mathrm{SL}_2(q)$ embeds absolutely irreducibly into $\Omega_5^\circ(q)$ by the module $S^4(W)$ with highest weight 4; hence by Lemma 4.1.21 the $\mathrm{SL}_2(q)$-module obtained as the restriction of the 20-dimensional module $V_4 \otimes V_5$ has highest weight at most 7, and thus we have the same restriction on the highest weight of the $\mathrm{SL}_2(q)$-module obtained as a restriction of the 16-dimensional submodule of $V_4 \otimes V_5$. The module corresponding to the 16-dimensional $\mathcal{S}_2^*$-candidate subgroup $\mathrm{SL}_2(q)$ of Ω is $S^{15}(W)$ with highest weight 15, and hence these are not the same representation. Hence there is no containment of $\mathrm{SL}_2(q)$ inside $\mathrm{Sp}_4(q)$ when considered as $\mathcal{S}_2^*$-candidate subgroups of Ω. $\square$

Proposition 4.5.5. *Let $\Omega = \Omega_{16}^+(q)$. Then we have the following containments:*

(i) $\mathrm{L}_2(q^2).2 < \mathrm{S}_4(q^2).2$ *for all q, and the normaliser of $\mathrm{L}_2(q^2).2$ is not maximal in any almost simple extension of Ω contained in $\mathrm{CGO}_{16}^+(q)$.*

(ii) $\mathrm{S}_4(q^2).2 < \mathrm{Sp}_8(q)$ *when q is even, and the normaliser of $\mathrm{S}_4(q^2).2$ is not maximal in any almost simple extension of Ω contained in $\mathrm{CGO}_{16}^+(q)$.*

The normaliser of every other $\mathcal{S}_2^*$-candidate subgroup of Ω is $\mathcal{S}_2^*$-maximal in Ω and its almost simple extensions contained in $\mathrm{CGO}_{16}^+(q)$.

Proof. From Theorem 4.2.1 we see that the $\mathcal{S}_2^*$-candidate subgroups are

(i) $\mathrm{L}_2(q^4).4$ for any prime power $q = p^e$.

(ii) $\mathrm{L}_2(q^2).2$ for $p \neq 2, 3$.

(iii) $\mathrm{Sp}_4(q)$ for $p = 2$ (although from Proposition 4.3.22 this is not $\mathcal{S}_2^*$-maximal).

(iv) $\mathrm{S}_4(q^2).2$ for any prime power.

(v) $2^{\cdot}\Omega_9^\circ(q)$ for q odd.

(vi) $\mathrm{Sp}_8(q)$ for q even.

We first consider $\mathrm{L}_2(q^2).2$. The corresponding module is $V_4 \otimes V_4^\sigma$, where V_4 denotes the symmetric cube of the natural representation of $\mathrm{L}_2(q^2)$. From Proposition 4.3.6, the action group of such a representation is contained in $\mathrm{Sp}_4(q^2)$. Since the module of the $\mathcal{S}_2^*$-candidate subgroup $\mathrm{Sp}_4(q^2)$ is $V \otimes V^\sigma$, for V the natural module of $\mathrm{Sp}_4(q^2)$, we have a containment of $\mathrm{L}_2(q^2) < \mathrm{Sp}_4(q^2)$. Since the restriction of the field automorphism σ_G of $\mathrm{Sp}_4(q^2)$ restricts to the corresponding field automorphism of $\mathrm{L}_2(q^2)$, we have a containment inside Ω of $\mathrm{L}_2(q^2).2 < \mathrm{Sp}_4(q^2).2$. From the tables the class stabilisers of both representations inside $\mathrm{CGO}_{16}^+(q)$ are $\langle \delta_\Omega \rangle$, with δ_Ω inducing $\delta_{\mathrm{L}_2(q^2)}$ and $\delta_{\mathrm{Sp}_4(q^2)}$ on the respective groups. Since the class stabilisers are the same there are no type-1 novelties. Further, from Proposition 4.3.6 it follows that $\delta_{\mathrm{Sp}_4(q^2)}$ induces $\delta_{\mathrm{L}_2(q^2)}$, thus ruling out the possibility of any type-2 novelties. Hence the normaliser of $\mathrm{L}_2(q^2).2$ is not maximal in any almost simple extension of Ω.

We next consider extensions of $\Omega_9^\circ(q)$ for odd q. We can rule out containments of any $\mathcal{S}_2^*$-candidate subgroup in any $\mathcal{C}_1$-subgroup of $2^{\cdot}\Omega_9^\circ(q)$ except $2^{\cdot}\Omega_8^\pm(q).2$ by

Corollary 4.5.3. For these groups, [41] shows there are no irreducible 16-dimensional representations of $\Omega_8^{\pm}(q)$, and since the nonabelian composition factor of the $\mathcal{S}_2^*$-candidate subgroup would have to be contained in $\Omega_8^{\pm}(q)$ by Lemma 4.5.1, it follows from Lemma 4.5.2 that there are no containments in $\mathcal{C}_1$-subgroups. This is enough to eliminate any possible containments of $\mathrm{S}_4(q^2)$ and $\mathrm{L}_2(q^4)$ inside $\Omega_9^{\circ}(q)$ when q is odd, since all the remaining maximal subgroups of $\Omega_9^{\circ}(q)$ are too small to contain either of these candidates.

For containments $\mathrm{L}_2(q^4).4$ inside $\mathrm{S}_4(q^2)$, analysis of the maximal subgroups of $\mathrm{S}_4(q^2)$ show that the only maximal subgroup which contains $\mathrm{L}_2(q^4)$ is the $\mathcal{C}_3$-subgroup $\mathrm{S}_2(q^4):2 \cong \mathrm{L}_2(q^4):2$, which clearly cannot contain $\mathrm{L}_2(q^4).4$. This covers all possible containments when q is odd.

When q is even, we again can rule out containments of groups in many of the maximal subgroups of $\mathrm{Sp}_8(q)$ by Corollary 4.5.3. The only groups we cannot rule out using this are the $\mathcal{C}_8$-subgroups $\mathrm{SO}_8^{\pm}(q)$ (which we can rule out using similar arguments to the q odd case) and the $\mathcal{C}_3$-subgroup $\mathrm{Sp}_4(q^2).2$. As in the previous paragraph we can rule out containments of $\mathrm{L}_2(q^4).4$ in this subgroup.

There is an abstract containment of the $\mathcal{S}_2^*$-subgroup $\mathrm{S}_4(q^2).2$ inside $\mathrm{Sp}_8(q)$, contained in the $\mathcal{C}_3$-subgroup. Computations in `s2cont` show that the subgroup $\mathrm{S}_4(2^2)$ of the spin representation $\mathrm{Spin}_9^{\circ}(2)$ is an absolutely irreducible subgroup of $\Omega_{16}^{+}(2)$. For q a larger power of 2, suppose that $\mathrm{S}_4(q^2) < \Omega_{16}^{+}(q)$ is reducible. Then we also have that $\mathrm{S}_4(2^2) < \Omega_{16}^{+}(q)$ is reducible, contradicting absolute irreducibility of $\mathrm{S}_4(2^2)$. Hence $\mathrm{S}_4(q^2) < \Omega_{16}^{+}(q)$ is irreducible. A similar argument shows that this embedding is in fact absolutely irreducible. It then follows from [41] and Theorem 4.1.28 that this module must be $V \otimes V^{\sigma}$ and hence we have a containment for all even q. For both groups, the class stabiliser in $\mathrm{CGO}_{16}^{+}(q)$ is trivial, and hence there are no novelty subgroups to consider. □

Proposition 4.5.6. *Let $\Omega = \mathrm{SL}_{16}(q)$. Then there are no containments between $\mathcal{S}_2^*$-candidate subgroups.*

Proof. From Theorem 4.2.1 we see that the $\mathcal{S}_2^*$-candidate subgroups are

(i) Extensions of $\mathrm{SL}_4(q)$ when $p = 3$.

(ii) Extensions of $\mathrm{L}_4(q^2)$ for any prime p.

(iii) Extensions of $(2, q-1)^{\cdot}\Omega_{10}^{+}(q)$ for any prime p.

There can be no containments between the first two families of groups; in characteristic 3 the latter of these groups takes the form $\mathrm{L}_4(q^2).2^2$, which has no subgroup isomorphic to $\mathrm{SL}_4(q)$.

For containments of $\mathrm{L}_4(q^2)$ in $(2, q-1)^{\cdot}\Omega^+_{10}(q)$, [8, Table 8.66] and Lagrange rules out containments in all maximal subgroups except $\mathrm{Sp}_8(q)$ for q even, and $\Omega^\circ_9(q)$ for q odd (both $\mathcal{C}_1$-subgroups). Both have only one irreducible 16-dimensional representation, namely the spin representation, which preserves an orthogonal form. Since none of the $\mathcal{S}_2^*$-candidate subgroups preserve an orthogonal form, it follows from Lemma 4.5.2 that there are no possible containments.

For containments of $\mathrm{SL}_4(q)$ in $(2, q-1)^{\cdot}\Omega^+_{10}(q)$, the above argument also allows us to rule out containments in the $\mathcal{C}_1$-candidate subgroups $\Omega^\circ_9(q).2$ and $\mathrm{Sp}_8(q)$, and Lemma 4.5.2 and Corollary 4.5.3 rule out any remaining containments.

□

Proposition 4.5.7. *Let $\Omega = \mathrm{SU}_{16}(q)$. Then there are no containments between $\mathcal{S}_2^*$-candidate subgroups.*

Proof. From Theorem 4.2.1 we see that the $\mathcal{S}_2^*$-candidate subgroups are

(i) $\mathrm{SU}_4(q).(q+1, 4)$ when $p = 3$.

(ii) Extensions of $\mathrm{L}_4(q^2)$ for any prime p.

(iii) Extensions of $(2, q-1)^{\cdot}\Omega^-_{10}(q)$ for any prime p.

Similarly to Proposition 4.5.6, $\mathrm{L}_4(q^2)$ has no subgroup isomorphic to $\mathrm{SU}_4(q)$ when $p = 3$, and we can rule out containments of either in $(2, q-1)^{\cdot}\Omega^+_{10}(q)$ by analysing maximal subgroups and applying Lemma 4.5.2 and Corollary 4.5.3. □

4.6 Summary

Theorem 4.6.1. *Let G and Ω be as in Remark 3.3.12, with $\Omega = \mathrm{SL}_{16}(q)$. Then representatives of the conjugacy classes of $\mathcal{S}_2^*$-maximal subgroups of G are as described in the list below:*

Proof. Direct from Theorem 4.2.1, Proposition 4.3.14, Proposition 4.3.17, Lemma 4.3.27, Lemma 4.4.7, Lemma 4.4.5 and Proposition 4.5.6. □

(i) S has nonabelian composition factor $\mathrm{L}_4(q)$ with $q = 3^e$. We list the possibilities below:

  (1) $S = \mathrm{SL}_4(q).2$ with e odd. The class stabiliser is $\langle \phi, \gamma \rangle$.

  (2) $S = \mathrm{SL}_4(q).2$ with $e \equiv 2 \mod 4$. The class stabiliser is $\langle \delta^4, \phi, \gamma \rangle$.

  (3) $S = \mathrm{SL}_4(q)$ with $e \equiv 0 \mod 4$. The class stabiliser is $\langle \delta^4, \phi, \gamma \rangle$.

(ii) S has nonabelian composition factor $\mathrm{L}_4(q^2)$ for any prime p. We list the possibilities below:

(1) $S = \mathrm{L}_4(q^2).2$ with $p = 2$. The class stabiliser is $\langle \phi, \gamma \rangle$.

(2) $S = 2^{\cdot}\mathrm{L}_4(q^2).(4 \times 2)$ with $q \equiv 3 \mod 4$. The class stabiliser is $\langle \phi, \gamma \rangle$.

(3) $S = 2^{\cdot}\mathrm{L}_4(q^2).(4 \times 2)$ with $q \equiv 5 \mod 8$. The class stabliser is $\langle \phi, \gamma \rangle$.

(4) $S = 2^{\cdot}\mathrm{L}_4(q^2).2^2$ with $q \equiv 9 \mod 16$. The class stabiliser is $\langle \delta^4, \phi, \gamma \rangle$.

(5) $S = 2^{\cdot}\mathrm{L}_4(q^2).2$ with $q \equiv 1 \mod 16$. The class stabiliser is $\langle \delta^4, \phi, \gamma \rangle$.

(iii) S has nonabelian composition factor $\mathrm{O}^+_{10}(q)$ for any prime p. We list the possibilities below:

(1) $S = \Omega^+_{10}(q)$ with $p = 2$. The class stabiliser is $\langle \phi, \gamma \rangle$.

(2) $S = 2^{\cdot}\Omega^+_{10}(q).2$ with $q \equiv 3 \mod 4$. The class stabiliser is $\langle \phi, \gamma \rangle$.

(3) $S = 2^{\cdot}\Omega^+_{10}(q).4$ with $q \equiv 5 \mod 8$. The class stabiliser is $\langle \phi, \gamma \rangle$.

(4) $S = 2^{\cdot}\Omega^+_{10}(q).2$ with $q \equiv 9 \mod 16$. The class stabiliser is $\langle \delta^4, \phi, \gamma \rangle$.

(5) $S = 2^{\cdot}\Omega^+_{10}(q)$ with $q \equiv 1 \mod 16$. The class stabiliser is $\langle \delta^4, \phi, \gamma \rangle$.

Theorem 4.6.2. *Let G and Ω be as in Remark 3.3.12, with $\Omega = \mathrm{SU}_{16}(q)$. Then representatives of the conjugacy classes of $\mathcal{S}_2^*$-maximal subgroups of G are as described in the list below:*

Proof. Direct from Theorem 4.2.1, Proposition 4.3.15, Proposition 4.3.18, Lemma 4.3.29, Lemma 4.4.8, Lemma 4.4.6 and Proposition 4.5.7. □

(i) $S = \mathrm{SU}_4(q).(q + 1, 4)$ with $q = 3^e$. The class stabiliser is $\langle \phi \rangle$.

(ii) S has nonabelian composition factor $\mathrm{L}_4(q^2)$ for any prime p. We list the possibilities below:

(1) $S = \mathrm{L}_4(q^2).2$ with $p = 2$. The class stabiliser is $\langle \phi \rangle$.

(2) $S = 2^{\cdot}\mathrm{L}_4(q^2).(4 \times 2)$ with $q \equiv 1 \mod 4$. The class stabiliser is $\langle \phi \rangle$.

(3) $S = 2^{\cdot}\mathrm{L}_4(q^2).(4 \times 2)$ with $q \equiv 3 \mod 8$. The class stabliser is $\langle \phi \rangle$.

(4) $S = 2^{\cdot}\mathrm{L}_4(q^2).2^2$ with $q \equiv 7 \mod 16$. The class stabiliser is $\langle \delta^4, \phi \rangle$.

(5) $S = 2^{\cdot}\mathrm{L}_4(q^2).2$ with $q \equiv 15 \mod 16$. The class stabiliser is $\langle \delta^4, \phi \rangle$.

(iii) S has nonabelian composition factor $\mathrm{O}^-_{10}(q)$ for any prime p. We list the possibilities below:

(1) $S = \Omega^-_{10}(q)$ with $p = 2$. The class stabiliser is $\langle \phi \rangle$.

(2) $S = 2{}^{\cdot}\Omega^-_{10}(q).2$ with $q \equiv 1 \mod 4$. The class stabiliser is $\langle \phi \rangle$.

(3) $S = 2{}^{\cdot}\Omega^-_{10}(q).4$ with $q \equiv 3 \mod 8$. The class stabiliser is $\langle \phi \rangle$.

(4) $S = 2{}^{\cdot}\Omega^-_{10}(q).2$ with $q \equiv 7 \mod 16$. The class stabiliser is $\langle \delta^4, \phi \rangle$.

(5) $S = 2{}^{\cdot}\Omega^-_{10}(q)$ with $q \equiv 15 \mod 16$. The class stabiliser is $\langle \delta^4, \phi \rangle$.

Theorem 4.6.3. *Let G and Ω be as in Remark 3.3.12, with $\Omega = \mathrm{Sp}_{16}(q)$. Then representatives of the conjugacy classes of $\mathcal{S}_2^*$-maximal subgroups of G are as described in the list below:*

Proof. Direct from Theorem 4.2.1, Corollary 4.3.7, Proposition 4.3.21, Lemma 4.4.9 and Proposition 4.5.4. □

(i) $S = \mathrm{SL}_2(q)$ with $p \geq 17$. The class stabiliser is $\langle \delta, \phi \rangle$.

(ii) $S = \mathrm{Sp}_4(q)$, with $p \neq 2, 5$. The class stabiliser is $\langle \delta, \phi \rangle$.

Theorem 4.6.4. *Let G and Ω be as in Remark 3.3.12, with $\Omega = \Omega^+_{16}(q)$. Assume Conjecture 2.3.3 holds. Then representatives of the conjugacy classes of $\mathcal{S}_2^*$-maximal subgroups of G are as described in the list below:*

Proof. Direct from Theorem 4.2.1, Proposition 4.3.8, Proposition 4.3.10, Proposition 4.3.22, Proposition 4.3.23, Lemma 4.3.28 and Proposition 4.5.5. □

(i) $S = \mathrm{L}_2(q^4).4$ for any prime p. If $p = 2$ then the class stabiliser in $\mathrm{CGO}^+_{16}(q)$ is trivial; otherwise the class stabiliser in $\mathrm{CGO}^+_{16}(q)$ is $\langle \delta \rangle$.

(ii) $S = \mathrm{S}_4(q^2).2$ with $p \neq 2$. The class stabiliser in $\mathrm{CGO}^+_{16}(q)$ is $\langle \delta \rangle$.

(iii) $S = 2{}^{\cdot}\Omega^\circ_9(q)$ with $p \neq 2$. The class stabiliser is $\langle \delta, \phi \rangle$.

(iv) $S = \mathrm{Sp}_8(q)$ with $p = 2$. The class stabiliser is $\langle \phi \rangle$.

Remark 4.6.5. Note that in Theorem 4.6.4 we only provide class stabilisers in $\mathrm{CGO}^+_{16}(q)$ rather than the full automorphism group of $\Omega^+_{16}(q)$ for some groups.

Theorem 4.6.6. *Let G and Ω be as in Remark 3.3.12, with $\Omega = \Omega^\circ_{17}(q)$. Then representatives of the conjugacy classes of $\mathcal{S}_2^*$-maximal subgroups of G are as described in the list below:*

Proof. Direct from Theorem 4.2.1 and Proposition 4.3.7. □

(i) $S = \mathrm{L}_2(q).2$ with $p \geq 17$. The class stabiliser is $\langle \phi \rangle$.

Theorem 4.6.7. *Let G and Ω be as in Remark 3.3.12, with $\Omega = \Omega_{16}^{-}(q)$, $\mathrm{SL}_{17}(q)$, or $\mathrm{SU}_{17}(q)$. Then there are no $\mathcal{S}_2^*$-maximal subgroups of G.*

Proof. Direct from Theorem 4.2.1. $\square$

Chapter 5

Construction of the spin groups

5.1 The theory of spin representations

5.1.1 Introduction

The *spin groups* are extensions of the simple group $O^{\epsilon}_m(q)$ by its full Schur multiplier. From [19, Table 2.2, p.39], we see that the Schur multiplier is $(2, q-1)$ when m is odd, $(2, q-1)^2$ when $m \equiv 0 \mod 4$ and $\epsilon = +$, and $(4, q^{\frac{m}{2}} - \epsilon 1)$ otherwise. Thus, the spin groups are double covers (when q is odd) of the groups $\Omega^{\epsilon}_m(q)$. Writing $m = 2n$ or $m = 2n+1$, the corresponding representations, which we call the *spin representations*, have dimension 2^n (see for instance [10, Theorem II.4.2 and Theorem II.5.2]).

When m is odd the spin representation is irreducible. When $m = 2n$ is even the spin representation is a direct sum of two 2^{n-1}-dimensional absolutely irreducible representations called the *half-spin representations*. We define the corresponding *half-spin groups* as the action groups of the half-spin representations. The half-spin groups are isomorphic to the spin groups except when $m \equiv 0 \mod 4$ and $\epsilon = +$, where the Schur multiplier is not cyclic and hence cannot centralise an absolutely irreducible representation; in this case the half-spin group is isomorphic to $(2, q-1)^{\cdot}O^{+}_m(q)$. In this case, when q is odd this group is not isomorphic to $\Omega^{+}_m(q)$, despite both being double covers of $O^{+}_m(q)$. To distinguish between these two, we write $\Omega^{+}_m(q) = (2, q-1)^{\cdot}_1 O^{+}_m(q)$, and $(2, q-1)^{\cdot}_2 O^{+}_m(q)$ for the half-spin representation.

This information is summarised in the table below:

Conditions	Centre of $\Omega^\epsilon_m(q)$	Centre of spin rep.	Centre of half-spin rep.
$m \equiv 0 \mod 4$, $\epsilon = +$, q odd	2	2×2	2
$m \equiv 2 \mod 4$, $\epsilon = +$, $q \equiv 1 \mod 4$	2	4	4
$m \equiv 2 \mod 4$, $\epsilon = +$, $q \equiv 3 \mod 4$	1	2	2
m even, $\epsilon = +$, q even	1	1	1
$m \equiv 0 \mod 4$, $\epsilon = -$, q odd	1	2	2
$m \equiv 2 \mod 4$, $\epsilon = -$, $q \equiv 1 \mod 4$	1	2	2
$m \equiv 2 \mod 4$, $\epsilon = -$, $q \equiv 3 \mod 4$	2	4	4
m even, $\epsilon = -$, q even	1	1	1
m odd, $\epsilon = \circ$, q odd	1	2	-
m odd, $\epsilon = \circ$, q even	1	1	-

For ease of notation we will write $\mathrm{Spin}^\epsilon_m(q)$ and $\mathrm{HSpin}^\epsilon_m(q)$ for the spin and half-spin groups respectively. Note that $\mathrm{HSpin}^\epsilon_{2n}(q)$ has two natural non-equivalent representations, although the groups themselves are isomorphic. If we need to distinguish between the two representations we will refer to $\mathrm{HSpin}^\epsilon_{2n}(q)$ and $\mathrm{HSpin}^\epsilon_{2n}(q)^\gamma$, since the graph automorphism γ of $\mathrm{HSpin}^\epsilon_{2n}(q)$ interchanges the two half-spin representations.

The usual way of constructing the spin representations is via Clifford algebras; see [59, Section 3.9] for an explicit construction of the spin group and the corresponding spin representations using this method. However our approach for this thesis will instead use highest weight theory to construct the representations.

Note that the numbering we use for roots of a Dynkin diagram is consistent with the ordering used by Magma. Other authors use the ordering as used in [16] which numbers the roots in the reverse order. This can lead to some confusion when referencing other results, especially in Section 5.5 where the numbering used by [53] and [49] is different to our numbering.

The general approach of this chapter, and some of the code, are due to [54]. Many of the computations in this chapter were performed in Magma [5].

5.1.2 Lie algebras

We will only provide a very bref summary of the pertinent information we will require about Lie algebras, and will assume some familiarity with the material. See for instance [9] for a more in-depth explanation. In particular, we will not define a Lie algebra.

Given a Lie algebra $\mathfrak{L}$ over $\mathbb{C}$, we can decompose it as $\mathfrak{L} = \mathfrak{H} \oplus \mathfrak{L}_{r_1} \oplus \cdots \oplus \mathfrak{L}_{r_m}$, where $\mathfrak{H}$ is a Cartan subalgebra of $\mathfrak{L}$, and the $\mathfrak{L}_{r_i}$ are 1-dimensional subspaces of $\mathfrak{L}$, known as the *root spaces* of $\mathfrak{L}$. We can also define maps $r_i : \mathfrak{H} \to \mathbb{C}$ based on how elements in $\mathfrak{H}$ scale the root space $\mathfrak{L}_{r_i}$, and these maps r_i are known as *roots*. This notion agrees with the notion of roots as introduced in Section 4.1.1.

It turns out that every root can be expressed as an integral linear combination of *fundamental roots* such that the integral coefficients for a given root are either all non-negative or all non-positive. For a given root α and a fixed system of fundamental roots, the *support* of α is those fundamental roots where the corresponding integral coefficients are nonzero. There is a well-known classification of all possible systems of fundamental roots via their *Dynkin diagram* (see [9, Section 3.4]), and to each Dynkin diagram we can associate a *Cartan matrix* (see [9, Section 3.6]).

5.1.3 The Curtis-Steinberg-Tits presentation

The Curtis-Steinberg-Tits presentation was discovered by Curtis [13] (and independently by Tits) adapting work done by Steinberg, and offers a presentation of central extensions of groups of Lie type. For our purposes, we will use a reduced presentation described in [3, Section 4.2].

Theorem 5.1.1 (Curtis-Steinberg-Tits presentation). *[3, Theorem 4.2] Fix Φ as the set of roots of a simple Lie algebra $\mathfrak{L}$ of rank $n \geq 2$, and Φ^+ a positive system of roots. For $1 \leq i < j \leq n$, let $\Phi_{i,j}$ denote the rank 2 subsystem spanned by the i-th and j-th fundamental roots, and set $\Psi = \bigcup_{i<j} \Phi_{i,j}$. Also set $\Upsilon_{i,j} = \{(\alpha, \beta) \in \Phi_{i,j} \times \Phi_{i,j} | \alpha \neq \pm\beta\}$ and $\Upsilon = \bigcup_{i<j} \Upsilon_{i,j}$. Let $\mathbb{F}_{p^e}$ be a finite field and $B = \{b_1, \ldots, b_e\}$ a basis of $\mathbb{F}_{p^e}$ as a $\mathbb{F}_p$-vector space. Then a presentation for a central extension of the simple group of Lie type G corresponding to the Lie algebra $\mathfrak{L}$ over $\mathbb{F}_{p^e}$ is given by generators $\{y_\alpha(b_t) : \alpha \in \Psi, t = 1, \ldots, e\}$ satisfying the following relations:*

$$\begin{aligned}
y_\alpha(b_t)^p &= 1 && \alpha \in \Psi, 1 \leq t \leq e \\
[y_\alpha(b_t), y_\alpha(b_u)] &= 1 && \alpha \in \Psi, 1 \leq t < u \leq e \\
[y_\alpha(b_t), y_\beta(b_u)] &= \prod_{i,j>0} y_{i\alpha+j\beta}(C_{i,j,\alpha,\beta} b_t^i b_u^j) && (\alpha, \beta) \in \Upsilon, 1 \leq t, u \leq n
\end{aligned}$$

where the $C_{i,j,\alpha,\beta}$ are the structure constants described below, and where for $\lambda = \sum_{t=1}^{e} \lambda_t b_t \in \mathbb{F}_{p^e}$ with $0 \leq \lambda_t < p$, we define $y_\alpha(\lambda) := \prod_{t=1}^{e} y_\alpha(b_t)^{\lambda_t}$.

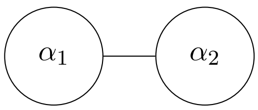
Remark 5.1.2. All terms in the product in the third set of relations in Theorem 5.1.1 commute, except when the group is $G_2(q)$. We will not be considering this group in this thesis, but details on how to define the product in this case can be found in [51].

We sketch the construction of the $C_{i,j,\alpha,\beta}$ - the full description of these constants can be found in [9, Theorem 5.2.2]. Since we will only be dealing with Dynkin diagrams of the form $\mathfrak{B}_n$ or $\mathfrak{D}_n$, we can assume that when α and β are fundamental roots at least one of i or j is equal to 1.

Let e_α and e_β denote two root vectors in the Lie algebra $\mathfrak{L}$ corresponding to roots α and β respectively, and suppose $\alpha+\beta$ is also a root. Then it follows that $[e_\alpha, e_\beta]$ is a scalar multiple of $e_{\alpha+\beta}$ where $[\ ,\ ]$ denotes the Lie bracket of $\mathfrak{L}$. Thus we can write $[e_\alpha, e_\beta] = N_{\alpha,\beta}e_{\alpha+\beta}$. (The integers $N_{\alpha,\beta}$ are known as the *structure constants* for the Lie algebra). It is the usual convention to set $N_{\alpha,\beta} = 0$ if $\alpha+\beta$ is not a root. With these structure constants, we then make the following definitions:

- $C_{i,1,\alpha,\beta} = \frac{1}{i!}\prod_{t=0}^{i-1} N_{\alpha,t\alpha+\beta}$.
- $C_{1,j,\alpha,\beta} = (-1)^j\frac{1}{j!}\prod_{t=0}^{j-1} N_{\beta,t\beta+\alpha}$.

Example 5.1.3. Let $\mathfrak{L}$ have Dynkin diagram $\mathfrak{A}_2$ as depicted below, and let α_1, α_2 denote the fundamental roots.



Since the two roots have a single edge connecting them, we have in the notation of Theorem 5.1.1

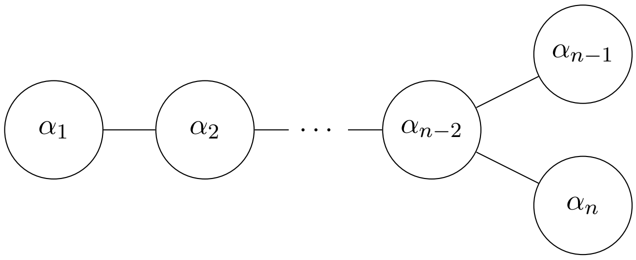
$$\Psi = \Phi_{1,2} = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2)\}.$$

We can consider the underlying Lie algebra directly, or use the Magma function `LieConstant_C`, to determine that the structure constants can be chosen as below, with $i = 1$ and $j = 2$:

$$\begin{aligned}
C_{1,1,\alpha_i,\alpha_j} &= 1 & C_{1,1,\alpha_j,\alpha_i} &= -1\\
C_{1,1,-\alpha_j,-\alpha_i} &= 1 & C_{1,1,-\alpha_i,-\alpha_j} &= -1\\
C_{1,1,\alpha_j,-\alpha_i-\alpha_j} &= 1 & C_{1,1,-\alpha_i-\alpha_j,\alpha_j} &= -1\\
C_{1,1,\alpha_i+\alpha_j,-\alpha_j} &= 1 & C_{1,1,-\alpha_j,\alpha_i+\alpha_j} &= -1\\
C_{1,1,-\alpha_i,\alpha_i+\alpha_j} &= 1 & C_{1,1,\alpha_i+\alpha_j,-\alpha_i} &= -1\\
C_{1,1,-\alpha_i-\alpha_j,\alpha_i} &= 1 & C_{1,1,\alpha_i,-\alpha_i-\alpha_j} &= -1
\end{aligned}$$

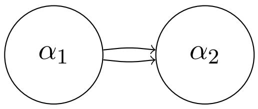
By convention we take the remaining structure constants to be 0.

Example 5.1.4. Let $\mathfrak{L}$ have Dynkin diagram $\mathfrak{D}_n$, as depicted below, and let $\alpha_1, \ldots, \alpha_n$ denote the fundamental roots.



Then for each pair of fundamental roots α_i, and α_j, there is either a single edge connecting them, or no edge. If there is no edge connecting α_i to α_j, then $\Phi_{i,j} = \{\pm\alpha_i, \pm\alpha_j\}$ and we do not need any structure constants to determine the relations in Theorem 5.1.1, since the subgroup generated by $\{y_\alpha(b) : \alpha \in \Phi_{i,j}, b \in \mathbb{F}_q\}$ will be isomorphic to $\mathrm{SL}_2(q) \times \mathrm{SL}_2(q)$ with corresponding Dynkin diagram $\mathfrak{A}_1 \times \mathfrak{A}_1$. If there is a single edge (and without loss of generality $i < j$) then we can use the structure constants as given in Example 5.1.3, since the subgroup generated by $\{y_\alpha(b) : \alpha \in \Phi_{i,j}, b \in \mathbb{F}_q\}$ will be isomorphic to $\mathrm{SL}_3(q)$ with corresponding Dynkin diagram $\mathfrak{A}_2$.

Example 5.1.5. Let $\mathfrak{L}$ have Dynkin diagram $\mathfrak{B}_2$ as depicted below. Let α_1, α_2 denote the fundamental roots.



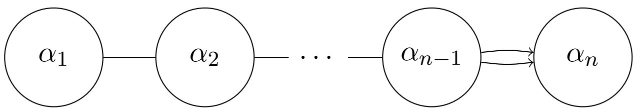
Due to the double edge connecting α_1 and α_2, we get that

$$\Psi = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2), \pm(\alpha_1 + 2\alpha_2)\}.$$

Taking $i = 1$ and $j = 2$ in the below table, we can derive the structure constants via Magma as in Example 5.1.3:

$$
\begin{aligned}
C_{1,1,\alpha_i,\alpha_j} &= 1 & C_{1,1,\alpha_j,\alpha_i} &= -1\\
C_{1,1,-\alpha_j,\alpha_i} &= 1 & C_{1,1,\alpha_i,-\alpha_j} &= -1\\
C_{1,1,\alpha_j,\alpha_i+\alpha_j} &= 2 & C_{1,1,\alpha_i+\alpha_j,\alpha_j} &= -2\\
C_{1,1,-\alpha_i-\alpha_j,-\alpha_j} &= 2 & C_{1,1,-\alpha_j,-\alpha_i-\alpha_j} &= -2\\
C_{1,1,\alpha_j,-\alpha_i-\alpha_j} &= 2 & C_{1,1,-\alpha_i-\alpha_j,\alpha_j} &= -2\\
C_{1,1,\alpha_i+\alpha_j,-\alpha_j} &= 2 & C_{1,1,-\alpha_j,\alpha_i+\alpha_j} &= -2\\
C_{1,1,\alpha_i+\alpha_j,-\alpha_i-2\alpha_j} &= 1 & C_{1,1,-\alpha_i-2\alpha_j,\alpha_i+\alpha_j} &= -1\\
C_{1,1,\alpha_i+2\alpha_j,-\alpha_i-\alpha_j} &= 1 & C_{1,1,-\alpha_i-\alpha_j,\alpha_i+2\alpha_j} &= -1\\
C_{1,1,-\alpha_i,\alpha_i+\alpha_j} &= 1 & C_{1,1,\alpha_i+\alpha_j,-\alpha_i} &= -1\\
C_{1,1,-\alpha_i-\alpha_j,\alpha_i} &= 1 & C_{1,1,\alpha_i,-\alpha_i-\alpha_j} &= -1\\
C_{1,1,-\alpha_j,\alpha_i+2\alpha_j} &= 1 & C_{1,1,\alpha_i+2\alpha_j,-\alpha_j} &= -1\\
C_{1,1,-\alpha_i-2\alpha_j,\alpha_j} &= 1 & C_{1,1,\alpha_j,-\alpha_i-2\alpha_j} &= -1\\
C_{2,1,\alpha_j,\alpha_i} &= 1 & C_{1,2,\alpha_i,\alpha_j} &= -1\\
C_{2,1,-\alpha_j,-\alpha_i} &= 1 & C_{1,2,-\alpha_i,-\alpha_j} &= -1\\
C_{2,1,\alpha_j,-\alpha_i-2\alpha_j} &= 1 & C_{1,2,-\alpha_i-2\alpha_j,\alpha_j} &= -1\\
C_{2,1,-\alpha_j,\alpha_i+2\alpha_j} &= 1 & C_{1,2,\alpha_i+2\alpha_j,-\alpha_j} &= -1\\
C_{1,2,\alpha_i,-\alpha_i-\alpha_j} &= 1 & C_{2,1,-\alpha_i-\alpha_j,-\alpha_i} &= -1\\
C_{1,2,\alpha_i,\alpha_i+\alpha_j} &= 1 & C_{2,1,\alpha_i+\alpha_j,-\alpha_i} &= -1\\
C_{1,2,\alpha_i+2\alpha_j,-\alpha_i-\alpha_j} &= 1 & C_{2,1,\alpha_i-\alpha_j,\alpha_i+2\alpha_j} &= -1\\
C_{1,2,-\alpha_1-2\alpha_2,\alpha_1+\alpha_2} &= 1 & C_{2,1,\alpha_i+\alpha_j,-\alpha_i-2\alpha_j} &= -1
\end{aligned}
$$

Example 5.1.6. Let $\mathfrak{L}$ have Dynkin diagram $\mathfrak{B}_n$ as depicted below. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ denote the fundamental roots.



Then, similarly to Example 5.1.4, we can use the structure constants from Examples 5.1.3 and 5.1.5 to find the structure constants, depending on the number of edges connecting α_i and α_j.

For the twisted groups, we require a separate presentation; the one we will use is also due to Steinberg (see for instance [20]), although we will again use a shortening of this presentation described in [3].

Theorem 5.1.7. *[3, Section 6.1] Let G be a twisted group of type ${}^2\mathfrak{A}_n(q)$ for $n > 1$ odd, ${}^2\mathfrak{D}_n(q)$ for $n \geq 4$, or ${}^2\mathfrak{E}_6(q)$, with associated twisted root systems of type $\mathfrak{C}_{\frac{n-1}{2}+1}$, $\mathfrak{D}_{n-1}$ or $\mathfrak{F}_4$ respectively. Define Ψ and Υ as in Theorem 5.1.1. Let $q = p^e$ and B denote a basis of $\mathbb{F}_{q^2}$ as a $\mathbb{F}_p$-vector space, with $B = \{b_1, \ldots, b_{2e}\}$ chosen such that $\{b_1, \ldots, b_e\}$ is a basis for $\mathbb{F}_q$ as an $\mathbb{F}_p$-vector space. Then we have a presentation of a central extension of G with the following form: it has generators $\{y_\alpha(b_t) : \alpha \in \Psi, t = 1, \ldots, k_\alpha e\}$ (where $k_\alpha = 1$ if α is a long root and $k_\alpha = 2$ if α is a short root), satisfying the following relations:*

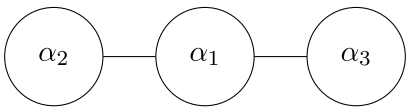
$$
\begin{array}{rlcl}
y_\alpha(b_t)^p &= 1 & \qquad & \alpha \in \Psi, 1 \leq t \leq k_\alpha e \\
{[y_\alpha(b_t), y_\alpha(b_u)]} &= 1 & \qquad & \alpha \in \Psi, 1 \leq t < u \leq k_\alpha e
\end{array}
$$

and, for $(\alpha, \beta) \in \Upsilon$, $1 \leq t \leq k_\alpha e$, $1 \leq u \leq k_\beta e$:

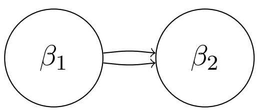
$$
\begin{array}{ll}
[y_\alpha(b_t), y_\beta(b_u)] = 1 & \alpha + \beta \notin \Psi \\
{[y_\alpha(b_t), y_\beta(b_u)]} = y_{\alpha+\beta}(\epsilon_{\alpha,\beta} b_t b_u) & \alpha, \beta, \alpha + \beta \textit{ all long or all short} \\
{[y_\alpha(b_t), y_\beta(b_u)]} = y_{\alpha+\beta}(\epsilon_{\alpha,\beta}(b_t \bar{b}_u + \bar{b}_t b_u)) & \alpha, \beta \textit{ short, } \alpha + \beta \textit{ long} \\
{[y_\alpha(b_t), y_\beta(b_u)]} = y_{\alpha+\beta}(\epsilon_{\alpha,\beta} b_t b_u) y_{\alpha+2\beta}(\eta_{\alpha,\beta} b_t b_u \bar{b}_u) & \alpha, \alpha + 2\beta \textit{ long, } \beta, \alpha + \beta \textit{ short}
\end{array}
$$

where the $\epsilon_{\alpha,\beta}$ and $\eta_{\alpha,\beta}$ take the value ± 1 and for $\lambda = \sum_{t=1}^{2e} \lambda_t b_t \in \mathbb{F}_{p^{2e}}$ with $0 \leq \lambda_t < p$, we define $y_\alpha(\lambda) := \prod_{t=1}^{2e} y_\alpha(b_t)^{\lambda_t}$.

Example 5.1.8. Let $\mathfrak{L}$ have Dynkin diagram $\mathfrak{A}_3 = \mathfrak{D}_3$ as depicted below. Let $\alpha_1, \alpha_2, \alpha_3$ denote the fundamental roots. Note the unusual numbering of the roots below - this is to keep the notation consistent when we generalise to $\mathfrak{D}_n$ later.



$\mathfrak{L}$ has a graph automorphism γ interchanging the roots α_2 and α_3. Suppose $\mathfrak{L}$ is a Lie algebra over $\mathbb{F}_{q^2}$ with field automorphism σ of order 2. The twisted group consists of those matrices which are fixed by the automorphism $\sigma\gamma$ (see [9] or [49] for more information on the construction of the twisted groups). The twisted group ${}^2\mathfrak{D}_3$ has underlying Lie algebra $\mathfrak{B}_2$, with fundamental roots β_1, β_2.



The generators $y_{\alpha_1}(b)$ of $\mathfrak{D}_3$ are always fixed by γ, and are fixed by σ when

$b \in \mathbb{F}_q^*$; hence we can take $y_{\beta_1}(b) = y_{\alpha_1}(b)$ for $b \in \mathbb{F}_q$. Similarly, $\sigma\gamma$ fixes $y_{\beta_2}(a) := y_{\alpha_2}(a)y_{\alpha_3}(a^q)$ for any $a \in \mathbb{F}_{q^2}^*$.

It turns out that we can derive the structure constants from the constants used in Example 5.1.5; see Remark 5.1.10.

Example 5.1.9. Let $\mathfrak{L}$ have Dynkin diagram $\mathfrak{D}_n$, with fundamental roots $\alpha_1, \ldots, \alpha_n$. Again, this has a graph automorphism γ of order 2 interchanging α_{n-1} and α_n, giving rise to the twisted group ${}^2\mathfrak{D}_n$ with underlying Dynkin diagram $\mathfrak{B}_{n-1}$ with fundamental roots $\beta_1, \ldots, \beta_{n-1}$. We can derive the corresponding generators as before:

$$\begin{aligned} y_{\beta_i}(b) &= y_{\alpha_i}(b) && \forall\, i \leq n-2, \forall\, b \in \mathbb{F}_q^* \\ y_{\beta_{n-1}}(r) &= y_{\alpha_{n-1}}(r)y_{\alpha_n}(r^q) && \forall\, r \in \mathbb{F}_{q^2}^* \end{aligned}$$

To determine the constants $\epsilon_{\alpha,\beta}$ and $\eta_{\alpha,\beta}$ from Theorem 5.1.7, we will again use results from smaller dimensions. For relations not involving $y_{\beta_{n-1}}(r)$, the generators involved are identical to the ones used in Theorem 5.1.1 applied to $\mathfrak{D}_n$ and so we can directly find $\epsilon_{\alpha,\beta}$ in this case. For relators involving $y_{\beta_{n-1}}(r)$, the structure constants will be the same as those discussed in Example 5.1.8.

Remark 5.1.10. It follows from [52, Table 0A8] that there is an embedding of $\mathfrak{B}_{n-1}$ inside ${}^2\mathfrak{D}_n$. This is obtained by the restriction of the group of Lie type corresponding to ${}^2\mathfrak{D}_n$ (which is over $\mathbb{F}_{q^2}$) to elements over the field $\mathbb{F}_q$. In particular, it follows from [52, p. 165] that we can obtain the commutator relations for $\mathfrak{B}_{n-1}$ from the commutator relations from ${}^2\mathfrak{D}_n$. However, where the structure constants are concerned, the converse is also true; since the structure constants only depend on the roots and not the field, the structure constants for ${}^2\mathfrak{D}_n$ and $\mathfrak{B}_{n-1}$ must coincide. Hence, we can use the structure constants as discussed in Example 5.1.5 and Example 5.1.6 for the twisted groups.

5.1.4 Diagonal automorphisms of $\Omega_n^\epsilon(q)$

The diagonal automorphisms of a Lie algebra $\mathfrak{L}$ over a field $\mathbb{F}_q$ are diagonal matrices when considered as acting on the Chevalley basis. Below we sketch the general construction, following [9].

Definition 5.1.11. Let Φ be a root system, with the set of fundamental roots given by Π. Let $P = \mathbb{Z}\Phi$ be the set of all $\mathbb{Z}$-linear combinations of elements of elements of Φ, which is a free abelian group with basis Π. Then an *$\mathbb{F}_q$-character of P* is a homomorphism from the additive group P to the multiplicative group $\mathbb{F}_q^*$.

Note that an $\mathbb{F}_q$-character is uniquely defined by its action on the set Π. Let $\Pi = \{\alpha_1, \ldots, \alpha_m\}$, so that P is free abelian of rank m.

Given an $\mathbb{F}_q$-character χ, we can obtain an automorphism of the Lie algebra $\mathfrak{L}$, denoted $h(\chi)$ and described as follows. The character χ is uniquely determined by the values $\chi(\alpha_i) = \lambda_i$. To each fundamental root α_i and nonzero element t in the field, the corresponding element $x_{\alpha_i}^{(n)}(t)$ of the adjoint Chevalley group is mapped to $x_{\alpha_i}^{(n)}(\lambda_i t)$, and the negative root $x_{-\alpha_i}^{(n)}(t)$ is mapped to $x_{-\alpha_i}^{(n)}(\lambda_i^{-1} t)$. This defines the action of $h(\chi)$ on a generating set of the adjoint Chevalley group Ω. Denote the set of such maps of $\mathfrak{L}$ by $\hat{H}$.

It follows from [9, p. 200] that all such maps give rise to automorphisms of Ω. Such automorphisms are the *diagonal automorphisms*, and this notation agrees with the notion of diagonal automorphisms as outer automorphisms of classical groups. The automorphisms $h(\chi)$ commute with diagonal elements of Ω.

Automorphisms in $\hat{H}$ may be inner or outer, and we will now establish conditions which determine precisely which elements induce inner or outer automorphisms of Ω.

Let $A = (A_{ij})$ denote the Cartan matrix relating to the Dynkin diagram of $\mathfrak{L}$, and construct Q as a free abelian group of rank m with basis β_i, where the β_i are fundamental weights and satisfy the relations $\alpha_i = \sum_{j=1}^{m} A_{ji}\beta_j$. Thus $\alpha_i \in Q$ for all i, and so P is a subgroup of Q, of index $\det A$. In particular, any $\mathbb{F}_q$-character of Q can be considered as an $\mathbb{F}_q$-character of P by restriction; however the converse is not true.

Theorem 5.1.12. *[9, Theorem 7.1.1] The elements $h(\chi)$ which lie inside the quasi-simple group Ω are precisely those where χ is an $\mathbb{F}_q$-character of P which can be extended to an $\mathbb{F}_q$-character of Q.*

For twisted groups of Lie type, we consider the group $\hat{\Omega}$ embedded into the corresponding untwisted group of Lie type Ω over $\mathbb{F}_{q^2}$. We will assume that the roots of the Lie algebra associated with Ω all have the same length (so the Lie algebra of Ω is not $\mathfrak{B}_2$, $\mathfrak{G}_2$ or $\mathfrak{F}_4$). Let $\alpha_1, \ldots, \alpha_k$ denote the roots of Ω, and τ the permutation of the roots inducing the twist of $\hat{\Omega}$. Write $\overline{\alpha_i}$ for α_{i^τ}, and for $f \in \mathbb{F}_{q^2}$ write $\bar{f} = f^q$. Define h, χ, P and Q as before for Ω.

Definition 5.1.13. The $\mathbb{F}_q$-character χ of P or Q is *self-conjugate* if $\chi(\overline{\alpha}_i) = \overline{\chi(\alpha_i)}$ for every fundamental root α_i.

Theorem 5.1.14. *[9, Theorem 13.7.2] The elements $h(\chi)$ which lie inside the quasi-simple group $\hat{\Omega}$ are precisely those where χ is a self-conjugate $\mathbb{F}_{q^2}$-character of P which can be extended to a self-conjugate $\mathbb{F}_{q^2}$-character of Q.*

5.2 Constructing the half-spin representation $\mathrm{HSpin}_{2n}^{+}(q)$

In this section, we will present generators for the half-spin representations $\mathrm{HSpin}_{2n}^{+}(q)$ in 2^{n-1} dimensions. We will first provide an inductive list of generators for the group, before proving their correctness using the Curtis-Steinberg-Tits presentation. We will then confirm that this is an irreducible representation of the half-spin group, and conclude by finding explicit generators for the diagonal automorphisms.

Note that we will only be considering the half-spin representations in this section. One can obtain the spin representation by taking the direct sum of the two half-spin representations.

5.2.1 Notation

In Theorem 5.1.1, we used the notation $y_\alpha(b)$ to denote a family of generators of the Chevalley group indexed by $b \in \mathbb{F}_q^*$, where α is a root which is the sum of multiples of at most two fundamental roots.

For $\mathfrak{D}_n$, this results in $4n-2$ families of generators, given by the fundamental roots $\alpha_1, \ldots, \alpha_n$ and the roots $\alpha_1+\alpha_2$, $\alpha_2+\alpha_3$, $\ldots$, $\alpha_{n-2}+\alpha_{n-1}$, $\alpha_{n-2}+\alpha_n$, plus the corresponding negative roots. At various points we will be considering roots of $\mathfrak{D}_n$ for several different choices of n simultaneously. To avoid confusion, and to suppress indices, we will use $y_i^{(n)}(b)$ in place of $y_{\alpha_i}(b)$ to denote the generator of one of the half-spin representations of $\mathfrak{D}_n$ corresponding to the fundamental root α_i, and similarly we will use $y_{(i)+(j)}^{(n)}(b)$ in place of $y_{\alpha_i+\alpha_j}(b)$. When there is no danger of confusion we will suppress the brackets in the subscript.

Recall that we use $E_{i,j}$ to denote the matrix with a 1 in the (i,j)-th position and 0 everywhere else. Such matrices will be square, and the dimension will always be clear from the context.

Recall the notation introduced in Section 1.6.1 for the direct and anti-direct sums of matrices.

We are now able to present generators for the half-spin representation of $\mathfrak{D}_3$, and then for the half-spin representation of $\mathfrak{D}_n$ in terms of generators for the half-spin representation of $\mathfrak{D}_{n-1}$. We will only present the generators $y_i^{(n)}(b)$; we will always define $y_{-i}^{(n)}(b) = y_i^{(n)}(b)^T$, and can obtain $y_{(i)+(j)}^{(n)}(b)$ in terms of $y_i^{(n)}(b)$ and $y_j^{(n)}(b)$ from the relations in Theorem 5.1.1.

5.2.2 Generators

The generators of $\mathrm{HSpin}_{2n}^{+}(q)$ are given by the matrices below and their transposes, where b ranges over all of $\mathbb{F}_q^*$:

$n = 3$

$$\begin{aligned} y_1^{(3)}(b) &= I_4 + bE_{3,2}, \\ y_2^{(3)}(b) &= I_4 - bE_{2,1}, \\ y_3^{(3)}(b) &= I_4 + bE_{4,3}. \end{aligned}$$

$n > 3$

$$\begin{aligned} y_1^{(n)}(b) &= I_{2^{n-1}} + b \sum_{j=1}^{2^{n-3}} E_{2^{n-2}+j, 2^{n-3}+j}, \\ y_i^{(n)}(b) &= y_{i-1}^{(n-1)}(b) \oplus y_{i-1}^{(n-1)}(b) \text{ for } 2 \leq i \leq n-2, \\ y_{n-1}^{(n)}(b) &= y_{n-2}^{(n-1)}(b) \oplus y_{n-1}^{(n-1)}(-b), \\ y_n^{(n)}(b) &= y_{n-1}^{(n-1)}(b) \oplus y_{n-2}^{(n-1)}(-b). \end{aligned}$$

Remark 5.2.1.

- The other half-spin representation $\mathrm{HSpin}_{2n}^{+}(q)^{\gamma}$ is obtained by interchanging the roles of $y_{n-1}^{(n)}(b)$ and $y_n^{(n)}(b)$.
- It is enough to choose $b \in B$, for B a $\mathbb{F}_p$-basis of the vector space $\mathbb{F}_q$, since we will prove later that $y_i^{(n)}(a)y_i^{(n)}(b) = y_i^{(n)}(a+b)$.

5.2.3 Proof of correctness of the generators

The following lemma and its corollary are straightforward but will be used frequently.

Lemma 5.2.2. *Let* $A = I_d + a \sum_{(i,j)\in X} E_{i,j}$, $B = I_d + b \sum_{(k,l)\in Y} E_{k,l}$, *where* $X, Y \subset \{1, \ldots, d\}^2$, *with* $(i,i) \notin X \cup Y$ *for all* i. *Let* $Z_{A,B} = \{(i,j,l) | (i,j) \in X, (j,l) \in Y\}$. *Then*

$$AB = I_d + a \sum_{(i,j)\in X} E_{i,j} + b \sum_{(k,l)\in Y} E_{k,l} + ab \sum_{(i,j,l)\in Z_{A,B}} E_{i,l}.$$

Corollary 5.2.3. *Let* $A, B, X, Y, Z_{A,B}$ *be as in Lemma 5.2.2. Then* *(i) If $Z_{A,A}$ is empty (equivalently, if for every i, there exists at most one j such that $(i,j) \in X$ or $(j,i) \in X$ (but not both)), then $A^{-1} = I_d - a \sum_{(i,j)\in X} E_{i,j}$.*

(ii) $[A,B] = I_d + ab \sum_{(i,j,l)\in Z_{AB}} E_{i,l} - ab \sum_{(i,j,l)\in Z_{BA}} E_{i,l}$.

(iii) A and B commute if and only if, for every pair (i,l):

$$|\{j : (i,j,l) \in Z_{AB}\}| = |\{j : (i,j,l) \in Z_{BA}\}|.$$

The next lemma will be useful when proving the irreducibility of the representation.

Lemma 5.2.4. *[8, Lemma 1.8.11] Let $V_1, \ldots, V_k$ be nonisomorphic irreducible G-modules. Then the only nontrivial submodules of $\oplus_{i=1}^k V_i$ are $\oplus_{j\in J} V_i$ for $J \subset \{1,\ldots,k\}$.*

Theorem 5.2.5. *Let $n \geq 3$, and $q = p^e$ be a prime power. Let B denote a basis of $\mathbb{F}_q$, viewed as a $\mathbb{F}_p$-vector space. Then the group generated by $\{y_i^{(n)}(b) : b \in B\}$ and $\{y_i^{(n)}(b)^T : b \in B\}$ as given in Section 5.2.2 is $\mathrm{HSpin}_{2n}^+(q)$.*

We prove Theorem 5.2.5 using the following lemmas.

Lemma 5.2.6. *The matrices as given in Section 5.2.2 satisfy the relations of Theorem 5.1.1.*

Proof. Note that every generator $y_i^{(n)}(b)$ satisfies the conditions of Lemma 5.2.2. In particular, each row and each column has at most two nonzero entries, one of which is a 1 on the diagonal, and the other (if it exists) takes the value $\pm b$. Thus it follows from Lemma 5.2.2 that $y_i^{(n)}(a)y_i^{(n)}(b) = y_i^{(n)}(a+b)$ for every choice of i, a, b, since Z is empty. In particular, this means that $y_i^{(n)}(b)^k = y_i^{(n)}(kb)$, so that $y_i^{(n)}(b)^p = y_i^{(n)}(pb) = I_{2^{n-1}}$, and so the first set of relations in Theorem 5.1.1 hold for the fundamental roots. Also, since $y_i^{(n)}(a)y_i^{(n)}(b) = y_i^{(n)}(a+b) = y_i^{(n)}(b)y_i^{(n)}(a)$, the second set of relations also hold for the fundamental roots.

In the context of Theorem 5.1.1, we can determine explicitly the roots in $\Phi_{i,j}$. This consists only of $\{\pm\alpha_i, \pm\alpha_j\}$ unless $j = i+1$ for $1 \leq i \leq n-2$ or $i = n-2, j = n$, in which case $\Phi_{i,j} = \{\pm\alpha_i, \pm\alpha_j, \pm(\alpha_i+\alpha_j)\}$. In the first case, there are four relations to check:

$$[y_i^{(n)}(a), y_j^{(n)}(b)] = 1 \qquad [y_{-j}^{(n)}(a), y_{-i}^{(n)}(b)] = 1$$
$$[y_i^{(n)}(a), y_{-j}^{(n)}(b)] = 1 \qquad [y_j^{(n)}(a), y_{-i}^{(n)}(b)] = 1$$

In the second case, the full list of relations to check, using the structure constants given in Example 5.1.4, are:

$$
\begin{array}{ll}
[y_i^{(n)}(a), y_j^{(n)}(b)] = y_{i+j}^{(n)}(ab) & [y_{-i}^{(n)}(a), y_j^{(n)}(b)] = 1 \\
[y_i^{(n)}(a), y_{-j}^{(n)}(b)] = 1 & [y_{-i}^{(n)}(a), y_{-j}^{(n)}(b)] = y_{-i-j}^{(n)}(-ab) \\
[y_i^{(n)}(a), y_{i+j}^{(n)}(b)] = 1 & [y_{-i}^{(n)}(a), y_{i+j}^{(n)}(b)] = y_j^{(n)}(ab) \\
[y_i^{(n)}(a), y_{-i-j}^{(n)}(b)] = y_{-j}^{(n)}(-ab) & [y_{-i}^{(n)}(a), y_{-i-j}^{(n)}(b)] = 1 \\
[y_j^{(n)}(a), y_i^{(n)}(b)] = y_{i+j}^{(n)}(-ab) & [y_{-j}^{(n)}(a), y_i^{(n)}(b)] = 1 \\
[y_j^{(n)}(a), y_{-i}^{(n)}(b)] = 1 & [y_{-j}^{(n)}(a), y_{-i}^{(n)}(b)] = y_{-i-j}^{(n)}(ab) \\
[y_j^{(n)}(a), y_{i+j}^{(n)}(b)] = 1 & [y_{-j}^{(n)}(a), y_{i+j}^{(n)}(b)] = y_i^{(n)}(-ab) \\
[y_j^{(n)}(a), y_{-i-j}^{(n)}(b)] = y_{-i}^{(n)}(ab) & [y_{-j}^{(n)}(a), y_{-i-j}^{(n)}(b)] = 1 \\
[y_{i+j}^{(n)}(a), y_j^{(n)}(b)] = 1 & [y_{-i-j}^{(n)}(a), y_j^{(n)}(b)] = y_{-i}^{(n)}(-ab) \\
[y_{i+j}^{(n)}(a), y_{-j}^{(n)}(b)] = y_i^{(n)}(ab) & [y_{-i-j}^{(n)}(a), y_{-j}^{(n)}(b)] = 1 \\
[y_{i+j}^{(n)}(a), y_i^{(n)}(b)] = 1 & [y_{-i-j}^{(n)}(a), y_i^{(n)}(b)] = y_{-j}^{(n)}(ab) \\
[y_{i+j}^{(n)}(a), y_{-i}^{(n)}(b)] = y_j^{(n)}(-ab) & [y_{-i-j}^{(n)}(a), y_{-i}^{(n)}(b)] = 1
\end{array}
$$

Note that some of these relations are superfluous. For instance, inverting the relation $[y_i^{(n)}(a), y_j^{(n)}(b)] = y_{i+j}^{(n)}(ab)$ gives the relation $[y_j^{(n)}(b), y_i^{(n)}(a)] = y_{i+j}^{(n)}(-ab)$. Thus we need only check half of these relations. Further, by transposing the relation $[y_i^{(n)}(a), y_j^{(n)}(b)] = y_{i+j}^{(n)}(ab)$, we obtain the relation $[y_{-j}^{(n)}(b), y_{-i}^{(n)}(a)] = y_{-i-j}^{(n)}(ab)$; thus we again only need to check the former. We list the reduced number of relations below; the left hand column records the relations to be checked, and the right hand column consists of the transposes of these relations.

$$
\begin{array}{ll}
[y_i^{(n)}(a), y_j^{(n)}(b)] = y_{i+j}^{(n)}(ab) & [y_{-j}^{(n)}(a), y_{-i}^{(n)}(b)] = y_{-i-j}^{(n)}(ab) \\
[y_i^{(n)}(a), y_{i+j}^{(n)}(b)] = 1 & [y_{-i-j}^{(n)}(a), y_{-i}^{(n)}(b)] = 1 \\
[y_{i+j}^{(n)}(a), y_j^{(n)}(b)] = 1 & [y_{-j}^{(n)}(a), y_{-i-j}^{(n)}(b)] = 1 \\
[y_i^{(n)}(a), y_{-j}^{(n)}(b)] = 1 & [y_j^{(n)}(a), y_{-i}^{(n)}(b)] = 1 \\
[y_{-i-j}^{(n)}(a), y_i^{(n)}(b)] = y_{-j}^{(n)}(ab) & [y_{-i}^{(n)}(a), y_{i+j}^{(n)}(b)] = y_j^{(n)}(ab) \\
[y_{i+j}^{(n)}(a), y_{-j}^{(n)}(b)] = y_i^{(n)}(ab) & [y_j^{(n)}(a), y_{-i-j}^{(n)}(b)] = y_{-i}^{(n)}(ab)
\end{array}
$$

We will take as our definition $y_{i+j}^{(n)}(b) := [y_i^{(n)}(1), y_j^{(n)}(b)]$ and $y_{-i-j}^{(n)}(b) := [y_{-j}^{(n)}(b), y_{-i}^{(n)}(1)]$ when the latter matrices do not commute; in other words, when

$i < n-1$ and $j = i+1$, or $i = n-2$ and $j = n$. Note with these definitions that $y_{i+j}^{(n)}(b)^T = y_{-i-j}^{(n)}(b)$. When $i < n-1$ we have

$$
\begin{aligned}
y_{i+(i+1)}^{(n)}(b) &= [y_i^{(n)}(1), y_{i+1}^{(n)}(b)] \\
&= [y_{i-1}^{(n-1)}(1) \oplus y_{i-1}^{(n-1)}(1), y_i^{(n-1)}(b) \oplus y_i^{(n-1)}(b)] \\
&= [y_{i-1}^{(n-1)}(1), y_i^{(n-1)}(b)] \oplus [y_{i-1}^{(n-1)}(1), y_i^{(n-1)}(b)] \\
&= y_{(i-1)+i}^{(n-1)}(b) \oplus y_{(i-1)+i}^{(n-1)}(b).
\end{aligned}
$$

We also have

$$
\begin{aligned}
y_{(n-2)+(n-1)}^{(n)}(b) &= [y_{n-2}^{(n)}(1), y_{n-1}^{(n)}(b)] \\
&= [y_{n-3}^{(n-1)}(1) \oplus y_{n-3}^{(n-1)}(1), y_{n-2}^{(n-1)}(b) \oplus y_{n-1}^{(n-1)}(-b)] \\
&= [y_{n-3}^{(n-1)}(1), y_{n-2}^{(n-1)}(b)] \oplus [y_{n-3}^{(n-1)}(1), y_{n-1}^{(n-1)}(-b)] \\
&= y_{(n-3)+(n-2)}^{(n-1)}(b) \oplus y_{(n-3)+(n-1)}^{(n-1)}(-b)
\end{aligned}
$$

and similarly $y_{(n-2)+n}^{(n)}(b) = y_{(n-3)+(n-1)}^{(n-1)}(b) \oplus y_{(n-3)+(n-2)}^{(n-1)}(-b)$. We can then check inductively for $n > 3$, or by hand for $n = 3$, that in all cases $y_{\pm(i+j)}^{(n)}(a) y_{\pm(i+j)}^{(n)}(b) = y_{\pm(i+j)}^{(n)}(a+b)$ and so the first and second sets of relations in Theorem 5.1.1 also hold for these generators.

We will now proceed to prove by induction that the third set of relations in Theorem 5.1.1 hold.

The base case $n = 3$ is a tedious but straightforward computation; thus, suppose $n > 3$. We will check all of the relations involving generators in $\Phi_{i,j}$.

Firstly, suppose $i = 1$. We first consider the case $j = 2$. By Lemma 5.2.2 we have:

$$
\begin{aligned}
y_1^{(n)}(b) &= I_{2^{n-1}} + b \sum_{i=1}^{2^{n-3}} E_{2^{n-2}+i, 2^{n-3}+i} \\
y_2^{(n)}(b) &= I_{2^{n-1}} + b \sum_{i=1}^{2^{n-4}} E_{2^{n-3}+i, 2^{n-4}+i} + b \sum_{i=1}^{2^{n-4}} E_{2^{n-2}+2^{n-3}+i, 2^{n-2}+2^{n-4}+i} \\
y_{1+2}^{(n)}(b) &= I_{2^{n-1}} + b \sum_{i=1}^{2^{n-4}} E_{2^{n-2}+i, 2^{n-4}+i} - b \sum_{i=1}^{2^{n-4}} E_{2^{n-2}+2^{n-3}+i, 2^{n-3}+2^{n-4}+i}
\end{aligned}
$$

By considering the rows and columns where each matrix has a nonzero entry, it follows from Corollary 5.2.3 that $y_{1+2}^{(n)}(a)$ commutes with both $y_1^{(n)}(b)$ and $y_2^{(n)}(b)$. It is easy to check the remaining relations by inspection and using Lemma 5.2.2, and we will only provide the details for one of the checks, namely for the relation $[y^{(n)}_{-1-2}(a), y^{(n)}_1(b)] = y^{(n)}_{-2}(ab)$. We have that

$$y^{(n)}_{-1-2}(a)y^{(n)}_1(b) = (y^{(n)}_{-1-2}(a) + y^{(n)}_1(b) - I_{2^{n-1}}) + ab\sum_{i=1}^{2^{n-4}} E_{2^{n-4}+i,2^{n-3}+i} \text{ ; and}$$

$$y^{(n)}_1(b)y^{(n)}_{-1-2}(a) = (y^{(n)}_{-1-2}(a) + y^{(n)}_1(b) - I_{2^{n-1}}) - ab\sum_{i=1}^{2^{n-4}} E_{2^{n-2}+2^{n-4}+i,2^{n-2}+2^{n-3}+i}$$

hence

$$\begin{aligned}[y^{(n)}_{-1-2}(a), y^{(n)}_1(b)] &= I_{2^{n-1}} + ab\sum_{i=1}^{2^{n-4}} E_{2^{n-4}+i,2^{n-3}+i} + ab\sum_{i=1}^{2^{n-4}} E_{2^{n-2}+2^{n-4}+i,2^{n-2}+2^{n-3}+i} \\ &= y^{(n)}_2(ab)^T = y^{(n)}_{-2}(ab).\end{aligned}$$

Next, we will show a sufficient condition for $y^{(n)}_1(b)$ to commute with a matrix g satisfying the conditions of Lemma 5.2.2. Suppose we can write $g = g_A \oplus g_B \oplus g_C \oplus g_D$. Let u^k_r (for $k \in \{A, B, C, D\}$) denote the column of the unique nonzero entry in the r-th row of g_k away from the diagonal (considering r modulo 2^{n-3}), or $u^k_r = 0$ if no such entry exists. Take a pair (r, s) with $1 \leq r, s \leq 2^{n-1}$ and $r \neq s$. Suppose that we can find a t such that the (r, t)-th entry of $y^{(n)}_1(a)$ is nonzero, as is the (t, s)-th entry of g. Then by the definition of $y^{(n)}_1(a)$ we must have $r \in \{2^{n-2} + 1, \ldots, 2^{n-2} + 2^{n-3}\}$ and $t = r - 2^{n-3}$. It follows that $s = u^B_r + 2^{n-3}$, and in particular we require $u^B_r \neq 0$.

Conversely, suppose that there exists a t such that the (r, t)-th entry of g is nonzero, as is the (t, s)-th entry of $y^{(n)}_1(a)$. For t to exist we require $u^k_r \neq 0$ for some k, and then $t = u^k_r + \left\lfloor \frac{r-1}{2^{n-3}} \right\rfloor 2^{n-3}$, the second term in this equation ensuring that the coordinate (r, t) lies in one of the blocks g_k of g. For the (t, s)-th entry of $y^{(n)}_1(a)$ to be nonzero, we require $t \in \{2^{n-2}+1, \ldots, 2^{n-2}+2^{n-3}\}$, which in particular implies that $r \in \{2^{n-2} + 1, \ldots, 2^{n-2} + 2^{n-3}\}$ so that $k = C$, and then $t = 2^{n-2} + u^C_r$ and $s = t - 2^{n-3} = 2^{n-3} + u^C_r$. Thus if $g_B = g_C$, in both cases we get the same restrictions on (r, s), and so by Corollary 5.2.3(3) the matrices commute.

Suppose $3 \leq j \leq n - 2$. Then we have $y^{(n)}_j(b) = y^{(n-2)}_{j-2}(b) \oplus y^{(n-2)}_{j-2}(b) \oplus y^{(n-2)}_{j-2}(b) \oplus y^{(n-2)}_{j-2}(b)$, and it follows from the above argument that $y^{(n)}_1(a)$ commutes with both $y^{(n)}_j(b)$ and $y^{(n)}_j(b)^T$.

For $j = n-1$ or $j = n$ we get that

$$y_{n-1}^{(n)}(b) = y_{n-2}^{(n-1)}(b) \oplus y_{n-1}^{(n-1)}(-b) = y_{n-3}^{(n-2)}(b) \oplus y_{n-2}^{(n-2)}(-b) \oplus y_{n-2}^{(n-2)}(-b) \oplus y_{n-3}^{(n-2)}(b).$$
$$y_{n}^{(n)}(b) = y_{n-1}^{(n-1)}(b) \oplus y_{n-2}^{(n-1)}(-b) = y_{n-2}^{(n-2)}(b) \oplus y_{n-3}^{(n-2)}(-b) \oplus y_{n-3}^{(n-2)}(-b) \oplus y_{n-2}^{(n-2)}(b).$$

In both cases the middle two summands are the same, so again these generators commute with $y_{\pm 1}^{(n)}(a)$. Hence we have shown that all the relations involving the generators $y_{\pm 1}^{(n)}(a)$ hold.

When $1 < i,j < n-1$, checking all of the relations hold follows immediately by induction, since $y_k^{(n)}(b) = y_{k-1}^{(n-1)}(b) \oplus y_{k-1}^{(n-1)}(b)$ for $k = i,j$, and $y_{i+j}^{(n)}(b) = y_{(i-1)+(j-1)}^{(n-1)}(b) \oplus y_{(i-1)+(j-1)}^{(n-1)}(b)$. If $1 < i < n-2$ and $j = n-1$ or $j = n$, or when $i = n-1$ and $j = n$, it again follows immediately by induction that all the generators involved commute.

Thus, the only remaining set of relations to check is when $i = n-2$ and $j = n-1$ or $j = n$. These again generally follow by straightforward induction, and we will only prove one of the relations here.

$$\begin{aligned}
&[y_{-(n-2)-(n-1)}^{(n)}(a), y_{n-2}^{(n)}(b)] \\
&= [y_{-(n-3)-(n-2)}^{(n-1)}(a) \oplus y_{-(n-3)-(n-1)}^{(n-1)}(-a), y_{n-3}^{(n-1)}(b) \oplus y_{n-3}^{(n-1)}(b)] \\
&= [y_{-(n-3)-(n-2)}^{(n-1)}(a), y_{n-3}^{(n-1)}(b)] \oplus [y_{-(n-3)-(n-1)}^{(n-1)}(-a), y_{n-3}^{(n-1)}(b)] \\
&= y_{-(n-2)}^{(n-1)}(ab) \oplus y_{-(n-1)}^{(n-1)}(ab) \\
&= y_{-(n-1)}^{(n)}(ab).
\end{aligned}$$

Hence all the relations of the CST presentation hold. □

Lemma 5.2.7. *The centre of the group generated by the matrices in Section 5.2.2 is the same as the centre of the half-spin representation as given in the table in Section 5.1.1.*

Proof. It is a standard result (see for instance [9, Lemma 6.4.4]) that the diagonal elements of the group are generated by elements of the form

$$h_i^{(n)}(t) = y_i^{(n)}(t) y_{-i}^{(n)}(-t^{-1}) y_i^{(n)}(t) y_i^{(n)}(-1) y_{-i}^{(n)}(1) y_i^{(n)}(-1).$$

It follows from similar proofs to those in Lemma 5.2.6 for the elements $y_i^{(n)}(t)$

that the following formulae hold:

$$\begin{aligned}
h_1^{(n)}(t) &= I_{2^{n-3}} \oplus t^{-1} I_{2^{n-3}} \oplus t I_{2^{n-3}} \oplus I_{2^{n-3}} \\
h_i^{(n)}(t) &= h_{i-1}^{(n-1)}(t) \oplus h_{i-1}^{(n-1)}(t) && \text{for } 2 \leq r \leq n-2 \\
h_{n-1}^{(n)}(t) &= h_{n-2}^{(n-1)}(t) \oplus h_{n-1}^{(n-1)}(-t) \\
h_n^{(n)}(t) &= h_{n-1}^{(n-1)}(t) \oplus h_{n-2}^{(n-1)}(-t)
\end{aligned}$$

Combining these with a direct computation which shows that

$$\begin{aligned}
h_1^{(3)}(t) &= \mathrm{diag}(1, t^{-1}, t, 1) \\
h_2^{(3)}(t) &= \mathrm{diag}(t^{-1}, t, 1, 1) \\
h_3^{(3)}(t) &= \mathrm{diag}(1, 1, t^{-1}, t)
\end{aligned}$$

we can explicitly construct the elements $h_i^{(n)}(t)$ in all cases.

A direct computation, or induction, gives that

$$h_{n-1}^{(n)}(t) h_n^{(n)}(t) = \mathrm{diag}(t^{-1}, t, t^{-1}, t, \ldots, t^{-1}, t).$$

When q is odd, we can take $t = t^{-1} = -1$ and obtain a central element of order 2.

We can obtain an additional central element precisely when n is odd and $q \equiv 1 \mod 4$. Since $q \equiv 1 \mod 4$, there exists an element $\lambda = \nu^{\frac{q-1}{4}} \in \mathbb{F}_q^*$ of order 4. Define

$$s^{(n)}(t) = h_1^{(n)}(-1) h_3^{(n)}(-1) \ldots h_{n-4}^{(n)}(-1) h_{n-2}^{(n)}(-1) h_{n-1}^{(n)}(t^{-1}) h_n^{(n)}(t).$$

We prove that $s^{(n)}(\lambda) = \lambda I_{2^{n-1}}$. As usual the case $s^{(3)}(\lambda)$ is easy to check. For $n \geq 5$ and $i \geq 3$ it follows immediately from the above calculations that

$$\begin{aligned}
h_i^{(n)}(t) &= h_{i-2}^{(n-2)}(t) \oplus h_{i-2}^{(n-2)}(t) \oplus h_{i-2}^{(n-2)}(t) \oplus h_{i-2}^{(n-2)}(t) \\
h_{n-1}^{(n)}(t) &= h_{n-3}^{(n-2)}(t) \oplus h_{n-2}^{(n-2)}(-t) \oplus h_{n-2}^{(n-2)}(-t) \oplus h_{n-3}^{(n-2)}(t) \\
h_n^{(n)}(t) &= h_{n-2}^{(n-2)}(t) \oplus h_{n-3}^{(n-2)}(-t) \oplus h_{n-3}^{(n-2)}(-t) \oplus h_{n-2}^{(n-2)}(t)
\end{aligned}$$

and hence

$$
\begin{aligned}
&s^{(n)}(\lambda)\\
&= h_1^{(n)}(-1)\left(h_3^{(n)}(-1)\dots h_{n-2}^{(n)}(-1)h_{n-1}^{(n)}(-\lambda)h_n^{(n)}(\lambda)\right)\\
&= (I_{2^{n-3}} \oplus -I_{2^{n-3}} \oplus -I_{2^{n-3}} \oplus I_{2^{n-3}})\\
&\qquad \cdot \left(s^{(n-2)}(\lambda) \oplus s^{(n-2)}(-\lambda) \oplus s^{(n-2)}(-\lambda) \oplus s^{(n-2)}(\lambda)\right)\\
&= (I_{2^{n-3}} \oplus -I_{2^{n-3}} \oplus -I_{2^{n-3}} \oplus I_{2^{n-3}})\,(\lambda I_{2^{n-3}} \oplus -\lambda I_{2^{n-3}} \oplus -\lambda I_{2^{n-3}} \oplus \lambda I_{2^{n-3}})\\
&= \lambda I_{2^{n-1}}.
\end{aligned}
$$

Hence when n is odd the centre of the representation equals the full Schur multiplier of $\mathrm{O}_{2n}^+(q)$, which is isomorphic to 2 when $q \equiv 3 \mod 4$ and 4 when $q \equiv 1 \mod 4$. When n is even, it follows from Lemma 5.2.8 below that the representation is absolutely irreducible. Since the Schur multiplier when n is even is 2×2, it is impossible for the centre to be larger than 2, as the representation is centralised only by scalars and so the centre will be cyclic. Thus we have a representation of $\mathrm{HSpin}_{2n}^+(q)$ in all cases. □

Lemma 5.2.8. *The module generated by the matrices in Section 5.2.2 is absolutely irreducible.*

Proof. We proceed by induction. When $n = 3$, we have that $\mathrm{O}_6^+(q) \cong \mathrm{L}_4(q)$, and we have the isomorphism $\mathrm{HSpin}_6^+(q) \cong \mathrm{SL}_4(q)$. In particular, the module we construct is a faithful 4-dimensional representation, and hence by [41] it must be the natural module of $\mathrm{SL}_4(q)$, which is absolutely irreducible. For $n > 3$, consider the subgroup generated by $y_{\pm i}^{(n)}(b)$ for $i = 2, \dots, n$. The corresponding group is a direct sum of the two half-spin representations of $\mathrm{HSpin}_{2n-2}^+(q)$, which by induction are absolutely irreducible and by [10, Theorem II.4.2] are non-isomorphic. Hence by Lemma 5.2.4 the only non-trivial submodules of this direct sum are the two half-spin representations, and neither of these are preserved by the action of the generator $y_1^{(n)}(b)$. Hence the half-spin representation of $\mathrm{HSpin}_{2n}^+(q)$ is absolutely irreducible. □

Proof of Theorem 5.2.5. It follows from Lemmas 5.2.6, 5.2.7 and 5.2.8 that the matrices generate an absolutely irreducible module of a group isomorphic to $\mathrm{HSpin}_{2n}^+(q)$. It follows from [33, Proposition 5.4.11] that the only absolutely irreducible representations of an extension of $\mathrm{O}_{2n}^+(q)$ in dimension 2^{n-1} are the half-spin representations, and so we are done.

Note that instead of using the results of Lemmas 5.2.7 and 5.2.8, we could show that the module is the absolutely irreducible half-spin representation by the direct computation of the elements $h_i^{(n)}(t)$ in the proof of Lemma 5.2.7. By considering the first entry of the diagonal, we see that this representation has highest weight $(0, 0, \ldots, 0, 1, 0)$, and the representation obtained by interchanging the roles of $y_n^{(n)}(b)$ and $y_{n-1}^{(n)}(b)$ has highest weight $(0, 0, \ldots, 0, 0, 1)$. By comparing the dimensions of these representations with the dimensions of the irreducible half-spin representations with these weights (as given for instance in [33, p.196]) and using Lemma 4.1.21 we obtain that both representations are absolutely irreducible. □

We next consider which classical groups the half-spin representations embed into, and the forms that they preserve (if any).

Lemma 5.2.9. *[33, Proposition 5.4.9] Let $G = \mathrm{HSpin}_{2n}^+(q)$. Then the half-spin representation embeds G into a classical group Ω as described below, and no smaller classical group in dimension 2^{n-1}.*

(i) If n is odd, then $\Omega = \mathrm{SL}_{2^{n-1}}(q)$.

(ii) If n is even and q is even, then $\Omega = \Omega^+_{2^{n-1}}(q)$.

(iii) If $n \equiv 0 \mod 4$ and q is odd, then $\Omega = \Omega^+_{2^{n-1}}(q)$.

(iv) If $n \equiv 2 \mod 4$ and q is odd, then $\Omega = \mathrm{Sp}_{2^{n-1}}(q)$.

Lemma 5.2.10. *Let $G = \mathrm{HSpin}_{2n}^+(q)$ denote the half-spin representation with n even, generated by the generators as given above. Then G preserves the form $f^{(n)}$ defined as follows:*

$$f^{(2)} = \mathrm{antidiag}(-1, 1).$$
$$f^{(n+2)} = -f^{(n)} \mathbin{\widehat{\oplus}} -f^{(n)} \mathbin{\widehat{\oplus}} f^{(n)} \mathbin{\widehat{\oplus}} f^{(n)}.$$

Proof. It is a direct computation to check that each generator preserves the appropriate form above. □

5.2.4 Automorphisms of the spin group

Recall the notation of Section 5.1.4. To determine the actions of the outer automorphisms on the spin group, we first need to determine which $\mathbb{F}_q$-characters of P can be extended to $\mathbb{F}_q$-characters of Q by Theorem 5.1.12. We establish this by determining the action of $\mathbb{F}_q$-characters of P on the standard generators of the natural representation of $\Omega^+_{2n}(q)$. From [47, Section 2.1] we obtain generators $x_i^{(n)}(b)$ for the natural representation of $\Omega^+_{2n}(q)$ (although the generators we use are transposed and in reverse order to those provided in [47]).

$$\begin{aligned}
x^{(n)}_i(b) &= I_{2n} + b(E_{i+1,i} - E_{2n+1-i,2n-i}) && \text{for } 1 \le i \le n-1,\\
x^{(n)}_n(b) &= I_{2n} + b(E_{n+2,n} - E_{n+1,n-1}).
\end{aligned}$$

We define $x^{(n)}_{-i}(b) = x^{(n)}_i(b)^T$.

Let χ be a $\mathbb{F}_q$-character of P, and suppose $\chi(\alpha_i) = \lambda_i$. Recall the notation $h(\chi)$ as introduced in Section 5.1.4. If $h(\chi)$ is induced by a diagonal matrix $\mathrm{diag}(a_1, \ldots, a_{2n})$, we obtain the following conditions on the a_i from the above generators:

$$\begin{aligned}
\lambda_i &= a^{-1}_{i+1}a_i = a^{-1}_{2n+1-i}a_{2n-i} && \text{for } 1 \le i \le n-1,\\
\lambda_n &= a^{-1}_{n+2}a_n = a^{-1}_{n+1}a_{n-1}.
\end{aligned}$$

We can obtain explicit formulae for the a_i by setting $a_{n+2} = t$ and solving, which gives:

$$\begin{aligned}
a_1 &= t\lambda_1\lambda_2\ldots\lambda_n,\\
a_2 &= t\lambda_2\lambda_3\ldots\lambda_n,\\
&\vdots\\
a_{n-1} &= t\lambda_{n-1}\lambda_n,\\
a_n &= t\lambda_n,\\
a_{n+1} &= t\lambda_{n-1},\\
a_{n+2} &= t,\\
a_{n+3} &= t\lambda^{-1}_{n-2},\\
a_{n+4} &= t\lambda^{-1}_{n-3}\lambda^{-1}_{n-2},\\
&\vdots\\
a_{2n} &= t\lambda^{-1}_1\lambda^{-1}_2\ldots\lambda^{-1}_{n-2}.
\end{aligned}$$

The standard generator of the automorphism $\delta \in \mathrm{CGO}^+_{2n}(q) \setminus \mathrm{GO}^+_{2n}(q)$ of $\Omega^+_{2n}(q)$ is given by $\mathrm{diag}(\nu, \ldots, \nu, 1, \ldots, 1)$, where ν is a primitive element of F^*_q. In the above, this corresponds to taking $\lambda_n = \nu$, $t = \lambda_1 = \lambda_2 = \ldots = \lambda_{n-1} = 1$; in other words the $\mathbb{F}_q$-character $(1, \ldots, 1, \nu)$ induces the δ automorphism of $\Omega^+_{2n}(q)$.

The standard generator of the automorphism $\delta' \in \mathrm{SO}^+_{2n}(q) \setminus \Omega^+_{2n}(q)$ (which is outer unless n is odd and $q \equiv 3 \mod 4$) is given by $\mathrm{diag}(1,\dots,1,\nu,\nu^{-1},1,\dots,1)$, which corresponds to taking $\lambda_n = \nu$, $\lambda_{n-1} = \nu^{-1}$, $t = \lambda_1 = \lambda_2 = \ldots = \lambda_{n-2} = 1$; thus the $\mathbb{F}_q$-character $(1,\dots,1,\nu^{-1},\nu)$ induces the δ' automorphism of $\Omega_{2n}^+(q)$.

Let $d_\alpha^{(n)}$ denote the matrix inducing the diagonal automorphism $\alpha \in \{\delta,\delta'\}$ on the half-spin representation $\mathrm{HSpin}_{2n}^+(q)$ generated by the generators described above. Recall that we obtain the other half-spin representation by interchanging the generators $y_{\pm(n-1)}^{(n)}(b)$ and $y_{\pm n}^{(n)}(b)$. This is same as the action of the graph automorphism γ, so we let $d_\alpha^{(n)\gamma}$ denote the matrix inducing the diagonal automorphism on the other half-spin representation. Note that these diagonal matrices are unique up to multiplication by a scalar; for consistency we will rescale so that the first diagonal entry of the matrix is 1.

Theorem 5.2.11. *Let ν denote a primitive element of $\mathbb{F}_q$, and $d_\alpha^{(n)}$ be as defined in the previous paragraph. Then*

(i) $d_\delta^{(3)} = \mathrm{diag}(1,1,1,\nu^{-1})$ *and* $d_\delta^{(3)\gamma} = \mathrm{diag}(1,\nu^{-1},\nu^{-1},\nu^{-1})$.

(ii) $d_\delta^{(n)} = d_\delta^{(n-1)} \oplus d_\delta^{(n-1)\gamma}$ *and* $d_\delta^{(n)\gamma} = d_\delta^{(n-1)\gamma} \oplus \nu^{-1} d_\delta^{(n-1)}$.

(iii) $d_{\delta'}^{(3)} = \mathrm{diag}(1,\nu,\nu,1)$ *and* $d_{\delta'}^{(3)\gamma} = \mathrm{diag}(1,\nu^{-1},\nu^{-1},1)$.

(iv) $d_{\delta'}^{(n)} = d_{\delta'}^{(n-1)} \oplus \nu d_{\delta'}^{(n-1)\gamma}$ *and* $d_{\delta'}^{(n)\gamma} = d_{\delta'}^{(n-1)\gamma} \oplus \nu^{-1} d_{\delta'}^{(n-1)}$.

Proof. It suffices to ensure that conjugation by the given matrices acts as expected on the generators.

For δ, this means that $y_n^{(n)}(a)^{d_\delta^{(n)}} = y_n^{(n)}(\nu a)$ and for $i < n$, $y_i^{(n)}(a)^{d_\delta^{(n)}} = y_i^{(n)}(a)$. It is easy to check this directly for $n = 3$. Inductively, it is clear from the definition that $y_i^{(n)}(a)^{d_\delta^{(n)}} = y_i^{(n)}(a)$ for $1 < i < n-1$ since both $d_\delta^{(n-1)}$ and $d_\delta^{(n-1)\gamma}$ commute with $y_{i-1}^{(n-1)}(a)$ by induction. Recall that $y_{n-1}^{(n)}(b) = y_{n-2}^{(n-1)}(b) \oplus y_{n-1}^{(n-1)}(-b)$ and $y_{n-1}^{(n-1)}(b) = y_{n-1}^{(n-1)}(b) \oplus y_{n-2}^{(n-1)}(-b)$. Since $d_\delta^{(n-1)}$ commutes with $y_{n-2}^{(n-1)}(b)$ and scales $y_{n-1}^{(n-1)}(b)$ to $y_{n-1}^{(n-1)}(\nu b)$, and $d_\delta^{(n-1)^\gamma}$ commutes with $y_{n-1}^{(n-1)}(b)$ and scales $y_{n-2}^{(n-1)}(b)$ to $y_{n-2}^{(n-1)}(\nu b)$, it follows that $d_\delta^{(n)}$ commutes with $y_{n-1}^{(n)}(b)$ and scales $y_n^{(n)}(b)$ to $y_n^{(n)}(\nu b)$ as required. It remains to check that $d_\delta^{(n)}$ commutes with $y_1^{(n)}(b)$. This is equivalent to ensuring that, if we write $d_\delta^{(n)} = m_1 \oplus m_2 \oplus m_3 \oplus m_4$ as a direct sum of four matrices of dimension 2^{n-3}, that $m_2 = m_3$. Inductively when $n \geq 5$ we have

$$d_\delta^{(n)} = d_\delta^{(n-1)} \oplus d_\delta^{(n-1)\gamma} = d_\delta^{(n-2)} \oplus d_\delta^{(n-2)\gamma} \oplus d_\delta^{(n-2)\gamma} \oplus \nu^{-1} d_\delta^{(n-2)}$$

so the matrices commute, and we can easily check that $d_\delta^{(4)}$ commutes with $y_1^{(4)}(b)$.

For $d_\delta^{(n)\gamma}$, a similar argument shows that if we write $d_\delta^{(n)\gamma} = d_\delta^{(n-1)\gamma} \oplus \lambda d_\delta^{(n-1)}$, then $d_\delta^{(n)\gamma}$ commutes with $y_i^{(n)}$ for $i \neq 1, n-1$, and that $y_{n-1}^{(n)}(b)^{d_\delta^{(n)\gamma}} = y_{n-1}^{(n)}(\nu b)$. To commute with $y_1^{(n)}(b)$, we require the middle two blocks to be equal; inductively for $n \geq 5$ we have

$$d_\delta^{(n)} = d_\delta^{(n-1)} \oplus \lambda d_\delta^{(n-1)\gamma} = d_\delta^{(n-2)} \oplus \lambda d_\delta^{(n-2)\gamma} \oplus \lambda d_\delta^{(n-2)\gamma} \oplus \lambda d_\delta^{(n-2)}$$

so the matrices commute, and when $n = 4$ we see that this condition is satisfied only when $\lambda = \nu^{-1}$. The proof for δ' is similar. □

Lemma 5.2.12. *Let $d_\alpha^{(n)}$ be as given in Theorem 5.2.11. Then*

(i) $\det d_{\delta'}^{(n)} = \nu^{2^{n-2}}$.

(ii) $\det d_{\delta'}^{(n)\gamma} = \nu^{-2^{n-2}}$.

(iii) $\det d_\delta^{(n)} = \nu^{2^{n-3}(2-n)}$.

(iv) $\det d_\delta^{(n)\gamma} = \nu^{-2^{n-3}n}$.

Proof. The formulae for the determinants of $d_{\delta'}^{(n)}$ and $d_{\delta'}^{(n)\gamma}$ are direct from Theorem 5.2.11.

Also from Theorem 5.2.11, if we set $\det d_\delta^{(n)} = \nu^{a_n}$ and $\det d_\delta^{(n)\gamma} = \nu^{b_n}$, then $a_3 = -1$, $b_3 = -3$ and the coefficients satisfy the recurrence relation

$$\begin{aligned} a_{n+1} &= a_n + b_n \\ b_{n+1} &= a_n + b_n - 2^{n-1} \end{aligned}$$

which we can reformulate into the single recurrence relation $a_{n+1} = 2a_n - 2^{n-2}$, which we can solve using standard recurrence relation techniques to obtain a formula for a_n, and then $b_n = a_n - 2^{n-2}$. □

5.2.5 Results

Lemma 5.2.13. *Let $\Omega = \mathrm{SL}_{2^{n-1}}(q)$ with n odd and $q = p^e$, let $G = (2, q-1)^{\cdot}\Omega_{2n}^+(q)$ be an $\mathcal{S}_2^*$-candidate subgroup of Ω, with G the action group of the half-spin representation, and let $S = N_\Omega(G)$. Then we have:*

(i) If q is even then $S = G$. We have a single Ω-class of subgroups isomorphic to S, with trivial class stabiliser in $\mathrm{GL}_{2^{n-1}}(q)$.

(ii) If $q \equiv 3 \mod 4$ and $n > 3$ then $S = G.2$ with the 2 automorphism induced by δ_G. We have two Ω-classes of subgroups isomorphic to S, with trivial class stabiliser in $\mathrm{GL}_{2^{n-1}}(q)$.

(iii) If $q \equiv 1 \mod 4$ and $q \not\equiv 1 \mod 2^{n-2}$, then $S = G.4$ with the 4 automorphism induced by δ_G. We have $(2^{n-1}, q-1)$ Ω-classes of subgroups isomorphic to S, with trivial class stabiliser in $\mathrm{GL}_{2^{n-1}}(q)$.

(iv) If $q \equiv 2^{n-2}+1 \mod 2^{n-1}$ then $S = G.2$, with the 2 automorphism induced by $\delta_G^2 = \delta'_G$. We have 2^{n-3} Ω-classes of subgroups isomorphic to S, with class stabiliser in $\mathrm{GL}_{2^{n-1}}(q)$ given by $\langle \delta_\Omega^{2^{n-3}} \rangle$, with $\delta_\Omega^{2^{n-3}}$ inducing δ_G, an automorphism of order 4.

(v) If $q \equiv 1 \mod 2^{n-1}$ then $S = G$. We have 2^{n-3} Ω-classes of subgroups isomorphic to S, with class stabiliser in $\mathrm{GL}_{2^{n-1}}(q)$ given by $\langle \delta_\Omega^{2^{n-3}} \rangle$, with $\delta_\Omega^{2^{n-3}}$ inducing δ_G, an automorphism of order 4.

In all cases, γ_Ω induces γ_G and ϕ_Ω induces ϕ_G.

Proof. The form preserved follows from Lemma 5.2.9.

Suppose q is odd. We first consider the δ_G automorphism of G. From Lemma 5.2.12 this is induced by a diagonal matrix with determinant $\nu^{-2^{n-3}(2-n)}$ where ν is a primitive element of $\mathbb{F}_q$. This is clearly a 2^{n-3}-power of an element of $\mathbb{F}_q$, and is a 2^{n-1}-power (and thus can be rescaled to lie in $\mathrm{SL}_{2^{n-1}}(q)$) if and only if $q \not\equiv 1 \mod 2^{n-2}$. If $q \equiv 1 \mod 2^{n-2}$ then δ_G is induced by a matrix with determinant a 2^{n-3}-power but not a 2^{n-2}-power, and hence is induced by (a conjugate of) $\delta_\Omega^{2^{n-3}}$.

We next consider δ'_G, which is nontrivial only when $q \equiv 1 \mod 4$. We can either perform similar computations to before on the determinant of the matrix inducing δ'_G, or we can notice that $\delta'_G = \delta_G^2$; both methods conclude that δ'_G is induced by a matrix over Ω if and only if $q \not\equiv 1 \mod 2^{n-1}$, and otherwise is induced by $\delta_\Omega^{2^{n-2}}$.

When q is even, neither G nor Ω have nontrivial diagonal outer automorphisms inside their respective conformal groups.

In all cases, the automorphism γ_G of G interchanges the two half-spin representations. The automorphism γ_Ω of Ω sends $y_i^{(n)}(a)$ to $y_{-i}^{(n)}(-a)$, and it is straightforward to check that γ_G is induced by γ_Ω followed by conjugation by the matrix $d_\gamma^{(n)}$, where:

$$d_\gamma^{(3)} = \mathrm{antidiag}(-1,-1,1,1),$$
$$d_\gamma^{(n+2)} = -d_\gamma^{(n)} \mathbin{\widehat{\oplus}} -d_\gamma^{(n)} \mathbin{\widehat{\oplus}} d_\gamma^{(n)} \mathbin{\widehat{\oplus}} d_\gamma^{(n)}.$$

The matrix $d_\gamma^{(n)}$ has determinant 1, so the automorphism induced by it is inner and so γ_G is induced by γ_Ω. That ϕ_G is induced by ϕ_Ω is direct from Corollary 4.4.2. $\square$

Lemma 5.2.14. *Let $\Omega = \mathrm{Sp}_{2^{n-1}}(q)$ with $n \equiv 2 \mod 4$ and $q = p^e$ with $p \neq 2$, let $G = 2{\dot{}}_2\mathrm{O}_{2n}^+(q)$ be an $\mathcal{S}_2^*$-candidate subgroup of Ω, with G the action group of the half-spin representation, and let $S = N_\Omega(G)$. Then we have $S = G.2$ with the 2 automorphism induced by δ_G. We have a single Ω-class of subgroups isomorphic to S, with class stabiliser in $\mathrm{CSp}_{2^{n-1}}(q)$ given by $\langle \delta_\Omega \rangle$ with δ_Ω inducing δ'_G. We also have that ϕ_Ω induces ϕ_G.*

Proof. The form preserved follows from Lemma 5.2.9, and by Lemma 5.2.10 it is antidiagonal.

We first consider the action of δ_G. The matrix $d_\delta^{(n)}$ lies in the conformal group of Ω, scaling the form by $\nu^{-\frac{n}{2}+1}$. Since $n \equiv 2 \mod 4$, it follows that $d_\delta^{(n)}$ rescales the form by a square, and hence can be rescaled to lie inside Ω. Thus δ_G can be realised by a matrix inside Ω.

We next consider δ'_G. Here $d_{\delta'}^{(n)}$ scales the form by ν, and so it follows that $d_{\delta'}^{(n)}$ cannot be rescaled to lie inside Ω, and so δ'_G is induced by the automorphism δ_Ω of Ω.

By Corollary 4.4.2, ϕ_Ω induces ϕ_G. $\square$

For the computations in the next lemma, we use the other half-spin representation to that used in the previous two lemmas, since these provide nicer automorphisms. We can move between the two half-spin representations by conjugating by the γ automorphism of $\mathrm{HSpin}_{2n}^+(q)$. This does not change the isomorphism class of the group.

Lemma 5.2.15. *Let $\Omega = \Omega_{2^{n-1}}^+(q)$ with $n \equiv 0 \mod 4$ and $q = p^e$ with $p \neq 2$, let $G = 2{\dot{}}_2\mathrm{O}_{2n}^+(q)$ be an $\mathcal{S}_2^*$-candidate subgroup of Ω, with G the action group of the half-spin representation, and let $S = N_\Omega(G)$. Then we have $S = G.2$ with the 2 automorphism induced by δ_G. We have four Ω-classes of subgroups isomorphic to S, with class stabiliser in $\mathrm{CGO}_{2^{n-1}}^+(q)$ given by $\langle \delta_\Omega \rangle$ with δ_Ω inducing δ'_G. We also have that ϕ_Ω induces ϕ_G.*

Proof. Again, the form preserved is from Lemma 5.2.9.

Considering δ_G first, as in Lemma 5.2.14 the matrix $d_\delta^{(n)\gamma}$ scales the form by $\nu^{-\frac{n}{2}}$; since $n \equiv 0 \mod 4$ we can scale $d_\delta^{(n)\gamma}$ by $\nu^{\frac{n}{4}}$ to obtain a matrix which preserves the form; further, by Lemma 5.2.12 such a matrix has determinant 1. It remains to compute the spinor norm. The matrices $I_{2^{n-1}} + \nu E_{i,i} + \nu E_{2^{n-1}+1-i, 2^{n-1}+1-i}$ are reflections in Ω, and hence we can determine the spinor norm of $\nu^{\frac{n}{4}} d_\delta^{(n)\gamma}$ by counting the powers of ν which appear in the first 2^{n-2} diagonal entries, and indeed since $\nu^{\frac{n}{4}} d_{\delta}^{(n)\gamma}$ is diagonal, the spinor norm will be determined by whether the product of the first 2^{n-2} diagonal entries is a square. By Theorem 5.2.11 and Lemma 5.2.12, this determinant is $\nu^{-2^{n-4}(n-1)}\nu^{n2^{n-3}}$. Since $n \neq 4$, this is a square, and so δ_G is inner.

We next consider δ'_G. The matrix $d_{\delta'}^{(n)\gamma}$ scales the form by ν^{-1} which is not a square in $\mathbb{F}_q$; hence $d_{\delta'}^{(n)\gamma}$ lies in the conformal group of Ω and thus is induced by either a conjugate of δ_Ω or a conjugate of $\delta_\Omega\gamma_\Omega$. Note however from the presentation that δ_Ω^2 is inner whereas $(\delta_\Omega\gamma_\Omega)^2 = \delta'_\Omega$, so we can determine which situation we are in by finding the spinor norm of $\nu(d_{\delta'}^{(n)\gamma})^2$, which preserves the form and has determinant 1. This matrix is diagonal with all entries equal to either ν or ν^{-1}; hence it is a product of 2^{n-2} reflections and thus has spinor norm 1. Hence δ'_G is induced by δ_Ω.

Since $\delta'_G = (\delta_G\gamma_G)^2$, if there was an automorphism α of Ω which induced γ_G, we would require α^2 to induce δ'_G, and hence $\alpha^2 = \delta_\Omega$. However since γ_G has order 2, α^2 would also have to be inner. Thus no such α exists, and thus no automorphism of Ω induces γ_G.

By Corollary 4.4.2 ϕ_Ω induces ϕ_G. $\square$

Note that when $n = 4$, the spin modules are images of the natural module under the triality automorphism, and stabilisers of the representation can be determined accordingly from stabilisers of the natural representation.

Lemma 5.2.16. *Let $\Omega = \Omega^+_{2^{n-1}}(q)$ with n even and $q = 2^e$, let $G = \Omega^+_{2n}(q)$ be an $\mathcal{S}_2^*$-candidate subgroup of Ω with G the action group of the half-spin representation, and let $S = N_\Omega(G)$. Then we have $S = G$. We have a single Ω-class of subgroups isomorphic to S, with trivial class stabiliser in $\mathrm{CGO}^+_{2^{n-1}}(q)$. We also have that ϕ_Ω induces ϕ_G.*

Proof. By Lemma 5.2.9, we have $G < \Omega$, with the outer automorphism group of Ω generated by γ_Ω and ϕ_Ω. By Corollary 4.4.2 ϕ_Ω induces ϕ_G, and since γ_G interchanges the two half-spin representations, it cannot be induced by any of the outer automorphisms of Ω. $\square$

5.3 Constructing the spin representation $\mathrm{Spin}^\circ_{2n+1}(q)$

5.3.1 Generators

The generators $z_i^{(n)}(b)$ of the spin representation $\mathrm{Spin}^\circ_{2n+1}(q)$ can be found from the generators of the half-spin representation of $\mathrm{HSpin}^+_{2n+2}(q)$.

$$\begin{aligned} z_i^{(n)}(b) &= y_i^{(n+1)}(b) && \text{for } 1 \le i \le n-1 \\ z_n^{(n)}(b) &= y_n^{(n+1)}(b) y_{n+1}^{(n+1)}(b) \end{aligned}$$

Equivalently, note that

$$\begin{aligned} &y_n^{(n+1)}(b) + y_{n+1}^{(n+1)}(b) - I_{2^{n-1}} - y_n^{(n+1)}(b) y_{n+1}^{(n+1)}(b) \\ &\quad = -(y_n^{(n+1)}(b) - I_{2^{n-1}})(y_{n+1}^{(n+1)}(b) - I_{2^{n-1}}) = 0 \end{aligned}$$

where the last equality follows by a direct computation. Hence we can also write

$$z_n^{(n)}(b) = y_n^{(n+1)}(b) + y_{n+1}^{(n+1)}(b) - I_{2^{n-1}}$$

and this definition may be more convenient in some places.

As in Section 5.2.2, the generators of the spin representation $\mathrm{Spin}^\circ_{2n+1}(q)$ are the elements $z_i^{(n)}(b)$ and their transposes.

5.3.2 Proof of correctness of the generators

The method we use to prove this is similar to that used in Theorem 5.2.5, and indeed many of the results will follow from the results proved there. One small technical complication comes from the fact that when there is a double edge connecting $z_i^{(n)}(b)$ and $z_j^{(n)}(b)$, the CST presentation involves two additional generators, $z_{i+j}^{(n)}(b)$ and $z_{i+2j}^{(n)}(b)$, and unlike the case when there is a single edge, formulae for these generators in terms of $z_i^{(n)}(b)$ and $z_j^{(n)}(b)$ do not follow immediately from the relations.

When there is a double edge between roots α_i and α_j, the list of relations from Theorem 5.1.1 is as follows:

$$
\begin{aligned}
&[z_i^{(n)}(a), z_j^{(n)}(b)] = z_{i+j}^{(n)}(ab) z_{i+2j}^{(n)}(-ab^2) && [z_{-i}^{(n)}(a), z_{-j}^{(n)}(b)] = z_{-i-j}^{(n)}(-ab) z_{-i-2j}^{(n)}(-ab^2) \\
&[z_i^{(n)}(a), z_{i+j}^{(n)}(b)] = 1 && [z_{-i-j}^{(n)}(a), z_{-i}^{(n)}(b)] = 1 \\
&[z_i^{(n)}(a), z_{i+2j}^{(n)}(b)] = 1 && [z_{-i}^{(n)}(a), z_{-i-2j}^{(n)}(b)] = 1 \\
&[z_i^{(n)}(a), z_{-j}^{(n)}(b)] = 1 && [z_j^{(n)}(a), z_{-i}^{(n)}(b)] = 1 \\
&[z_i^{(n)}(a), z_{-i-j}^{(n)}(b)] = z_{-j}^{(n)}(-ab) z_{-i-2j}^{(n)}(ab^2) && [z_{-i}^{(n)}(a), z_{i+j}^{(n)}(b)] = z_j^{(n)}(ab) z_{i+2j}^{(n)}(ab^2) \\
&[z_i^{(n)}(a), z_{-i-2j}^{(n)}(b)] = 1 && [z_{i+2j}^{(n)}(a), z_{-i}^{(n)}(b)] = 1 \\
&[z_j^{(n)}(a), z_{i+j}^{(n)}(b)] = z_{i+2j}^{(n)}(2ab) && [z_{-i-j}^{(n)}(a), z_{-j}^{(n)}(b)] = z_{-i-2j}^{(n)}(2ab) \\
&[z_j^{(n)}(a), z_{i+2j}^{(n)}(b)] = 1 && [z_{-i-2j}^{(n)}(a), z_{-j}^{(n)}(b)] = 1 \\
&[z_j^{(n)}(a), z_{-i-j}^{(n)}(b)] = z_{-i}^{(n)}(2ab) && [z_{i+j}^{(n)}(a), z_{-j}^{(n)}(b)] = z_i^{(n)}(2ab) \\
&[z_{-i-2j}^{(n)}(a), z_j^{(n)}(b)] = z_{-i-j}^{(n)}(ab) z_{-i}^{(n)}(-ab^2) && [z_{i+2j}^{(n)}(a), z_{-j}^{(n)}(b)] = z_{i+j}^{(n)}(-ab) z_i^{(n)}(-ab^2) \\
&[z_{i+j}^{(n)}(a), z_{i+2j}^{(n)}(b)] = 1 && [z_{-i-2j}^{(n)}(a), z_{-i-j}^{(n)}(b)] = 1 \\
&[z_{-i-2j}^{(n)}(a), z_{i+j}^{(n)}(b)] = z_{-j}^{(n)}(-ab) z_i^{(n)}(ab^2) && [z_{i+2j}^{(n)}(a), z_{-i-j}^{(n)}(b)] = z_j^{(n)}(ab) z_{-i}^{(n)}(ab^2)
\end{aligned}
$$

As before, we only list half of the relations; the other half are obtained by inverting these relations. The relations on the right hand side are also directly obtainable from those on the left hand side by transposing. Thus typically we will only need to check the relations on the left hand side.

Lemma 5.3.1. *Assume that the relations in Theorem 5.1.1 hold for relators $z_i^{(n)}(a)$ and $z_j^{(n)}(a)$ where there is a double edge between the corresponding root elements α_i and α_j. Suppose also that $z_\alpha^{(n)}(a)^{-1} = z_\alpha^{(n)}(-a)$ for any root α with support contained in $\{i, j\}$. Then:*

$$
\begin{aligned}
z_{i+2j}^{(n)}(a) &= z_{-j}^{(n)}(1) z_j^{(n)}(-1) z_i^{(n)}(-a) z_j^{(n)}(1) z_{-j}^{(n)}(-1) \\
z_{i+j}^{(n)}(a) &= z_i^{(n)}(-a) z_j^{(n)}(-1) z_i^{(n)}(a) z_j^{(n)}(1) z_{i+2j}^{(n)}(a).
\end{aligned}
$$

Proof. We will use some of the relations in Theorem 5.1.1; specifically:

$$
\begin{aligned}
[z_i^{(n)}(a), z_j^{(n)}(b)] &= z_{i+j}^{(n)}(ab) z_{i+2j}^{(n)}(-ab^2), \\
[z_{i+2j}^{(n)}(c), z_{-j}^{(n)}(d)] &= z_{i+j}^{(n)}(-cd) z_i^{(n)}(-cd^2).
\end{aligned}
$$

We invert the latter to get

$$
[z_{-j}^{(n)}(d), z_{i+2j}^{(n)}(c)] = z_{i+j}^{(n)}(cd) z_i^{(n)}(cd^2).
$$

Expanding and multiplying together, we get

$$z_{i+j}^{(n)}(ab)z_{i+2j}^{(n)}(-ab^2)z_{i+j}^{(n)}(cd)z_i^{(n)}(cd^2) = [z_i^{(n)}(a), z_j^{(n)}(b)][z_{-j}^{(n)}(d), z_{i+2j}^{(n)}(c)].$$

Note that from the relations in Theorem 5.1.1, all the terms on the left hand side commute; in particular, the $z_{i+j}^{(n)}$ terms will cancel if $cd = -ab$. Similarly, a $z_{i+2j}^{(n)}$ term from each side will cancel if $-ab^2 = c$. Assuming both of these conditions, we can rearrange to obtain:

$$z_i^{(n)}(cd^2) = z_i^{(n)}(-a)z_j^{(n)}(-b)z_i^{(n)}(a)z_j^{(n)}(b)z_{-j}^{(n)}(-d)z_{i+2j}^{(n)}(-c)z_{-j}^{(n)}(d)$$

and so

$$1 = z_i^{(n)}(-cd^2)z_i^{(n)}(-a)z_j^{(n)}(-b)z_i^{(n)}(a)z_j^{(n)}(b)z_{-j}^{(n)}(-d)z_{i+2j}^{(n)}(-c)z_{-j}^{(n)}(1).$$

This rearranges to obtain

$$z_{i+2j}^{(n)}(c) = z_{-j}^{(n)}(d)z_i^{(n)}(-cd^2)z_i^{(n)}(-a)z_j^{(n)}(-b)z_i^{(n)}(a)z_j^{(n)}(b)z_{-j}^{(n)}(-d).$$

Taking $b = d = 1$ and $a = -c$, we obtain the first relation (interchanging the roles of a and c). and our first formulae holds.

We can then use the relation $[z_i^{(n)}(a), z_j^{(n)}(b)] = z_{i+j}^{(n)}(ab)z_{i+2j}^{(n)}(-ab^2)$ to get

$$z_{i+j}^{(n)}(a) = [z_i^{(n)}(a), z_j^{(n)}(1)]z_{i+2j}^{(n)}(a).$$

□

Remark 5.3.2. In the proof of the above, we could also take $b = d = -1$ and $a = -c$, giving the alternate formula:

$$z_{i+2j}^{(n)}(a) = z_{-j}^{(n)}(-1)z_j^{(n)}(1)z_i^{(n)}(-a)z_j^{(n)}(-1)z_{-j}^{(n)}(1).$$

The next results hold specifically for the spin groups.

Lemma 5.3.3. *Let $z_i^{(n)}(b)$ be as defined in Section 5.3.1. If the relations in Theorem 5.1.1 hold, then the following formulae hold, for $n > 2$.*

(1) $z_{(1)+(2)}^{(2)}(b) = I_4 - bE_{3,1} - bE_{4,2}$.

(2) $z_{(1)+2(2)}^{(2)}(b) = I_4 - bE_{4,1}$.

(3) $z_i^{(n)}(b) = z_{i-1}^{(n-1)}(b) \oplus z_{i-1}^{(n-1)}(b)$ *for all i with $2 \leq i \leq n-1$.*

(4) $z_n^{(n)}(b) = z_{n-1}^{(n-1)}(b) \oplus z_{n-1}^{(n-1)}(-b)$.

(5) $z_{(n-1)+(n)}^{(n)}(b) = z_{(n-2)+(n-1)}^{(n-1)}(b) \oplus z_{(n-2)+(n-1)}^{(n-1)}(-b)$.

(6) $z_{(n-1)+2(n)}^{(n)}(b) = z_{(n-2)+2(n-1)}^{(n-1)}(b) \oplus z_{(n-2)+2(n-1)}^{(n-1)}(b)$.

Proof. The proofs of (1) and (2) are direct computations using Lemma 5.3.1 and the definitions in Section 5.3.1. (3) follows from Section 5.2.2. For (4) we have

$$\begin{aligned}
z_n^{(n)}(b) &= y_n^{(n+1)}(b) y_{n+1}^{(n+1)}(b) \\
&= \left(y_{n-1}^{(n)}(b) \oplus y_n^{(n)}(-b)\right)\left(y_n^{(n)}(b) \oplus y_{n-1}^{(n)}(-b)\right) \\
&= \left(y_{n-1}^{(n)}(b) y_n^{(n)}(b)\right) \oplus \left(y_n^{(n)}(-b) y_{n-1}^{(n)}(-b)\right) \\
&= z_{n-1}^{(n-1)}(b) \oplus z_{n-1}^{(n-1)}(-b).
\end{aligned}$$

From this and Lemma 5.3.1, we can derive (6):

$$\begin{aligned}
z_{(n-1)+2(n)}^{(n)}(b) &= z_{-n}^{(n)}(1) z_n^{(n)}(-1) z_{n-1}^{(n)}(-b) z_n^{(n)}(1) z_{-n}^{(n)}(-1) \\
&= (z_{-(n-1)}^{(n-1)}(1) z_{n-1}^{(n-1)}(-1) z_{n-2}^{(n-1)}(-b) z_{n-1}^{(n-1)}(1) z_{-(n-1)}^{(n-1)}(-1)) \\
&\quad \oplus (z_{-(n-1)}^{(n-1)}(-1) z_{n-1}^{(n-1)}(1) z_{n-2}^{(n-1)}(-b) z_{n-1}^{(n-1)}(-1) z_{-(n-1)}^{(n-1)}(1)) \\
&= z_{(n-2)+2(n-1)}^{(n-1)}(b) \oplus z_{(n-2)+2(n-1)}^{(n-1)}(b)
\end{aligned}$$

where the last line follows from Remark 5.3.2. Finally we prove (5):

$$\begin{aligned}
z_{(n-1)+(n)}^{(n)}(b) &= [z_{n-1}^{(n)}(b), z_n^{(n)}(1)] z_{(n-1)+2(n)}^{(n)}(b) \\
&= [z_{n-2}^{(n-1)}(b), z_{n-1}^{(n-1)}(1)] z_{(n-2)+2(n-1)}^{(n-1)}(b) \\
&\oplus [z_{n-2}^{(n-1)}(b), z_{n-1}^{(n-1)}(-1)] z_{(n-2)+2(n-1)}^{(n-1)}(b) \\
&= z_{(n-2)+(n-1)}^{(n-1)}(b) \oplus z_{(n-2)+(n-1)}^{(n-1)}(-b).
\end{aligned}$$

□

Hence, if the relations in Theorem 5.1.1 hold, it must follow that the generators $z_{(n-1)+(n)}^{(n)}(b)$ and $z_{(n-1)+2(n)}^{(n)}(b)$ must have the structure as described in Lemma 5.3.1 and Lemma 5.3.3. Thus we will use the results of Lemma 5.3.3 as our definitions of the generators $z_{(n-1)+(n)}^{(n)}(b)$ and $z_{(n-1)+2(n)}^{(n)}(b)$, and prove below that these definitions, plus the generators as given in Section 5.3.1, satisfy the conditions of Theorem 5.1.1.

Remark 5.3.4. We can perform similar computations using the corresponding relations for the negative roots to derive that in the spin groups, $z_{-i-j}^{(n)}(b) = z_{i+j}^{(n)}(b)^T$ and $z_{-i-2j}^{(n)}(b) = z_{i+2j}^{(n)}(b)^T$.

Theorem 5.3.5. *Let $n \geq 2$, and $q = p^e$ be a prime power. Let B denote a basis of $\mathbb{F}_q$ viewed as a $\mathbb{F}_p$-vector space. Then the group generated by $\{z_i^{(n)}(b) : b \in B\}$ and $\{z_i^{(n)}(b)^T : b \in B\}$ with the generators as given in Section 5.3.1 is $\mathrm{Spin}^{\circ}_{2n+1}(q)$.*

Proof. We again aim to show that the generators satisfy the conditions of Theorem 5.1.1. Any relations which do not involve the root α_n follow directly from Theorem 5.2.5.

From inspection and Lemma 5.2.2 we can see that $z_n^{(n)}(a)z_n^{(n)}(b) = z_n^{(n)}(a + b)$. From Lemma 5.2.2 and Lemma 5.3.3 we can conclude that the same holds for $z_{(n-1)+(n)}^{(n)}(a)$ and $z_{(n-1)+2(n)}^{(n)}(a)$, and thus similarly to Theorem 5.2.5 the first two sets of relations in Theorem 5.1.1 hold.

It also follows from the corresponding relations in Theorem 5.2.5 that $z_{\pm i}^{(n)}(a) = y_{\pm i}^{(n+1)}(a)$ commutes with $z_{\pm n}^{(n)}(b) = y_{\pm n}^{(n+1)}(b)y_{\pm(n+1)}^{(n+1)}(b)$ for any $i \leq n-1$.

It remains to check the relations involving the roots α_{n-1} and α_n. Again, these can be straightforwardly checked by hand or computer for $n = 2$, and we proceed by induction for larger n. The results for pairs of generators which commute follow immediately by induction and Lemma 5.3.3 (4)-(6). The remaining six relators also follow relatively straightforwardly. We provide only one of the computations; the rest are similar.

$$
\begin{aligned}
&[z_{n-1}^{(n)}(a), z_{-(n-1)-n}^{(n)}(b)] \\
&= [z_{n-2}^{(n-1)}(a) \oplus z_{n-2}^{(n-1)}(a), z_{-(n-2)-(n-1)}^{(n-1)}(b) \oplus z_{-(n-2)-(n-1)}^{(n-1)}(-b)] \\
&= [z_{n-2}^{(n-1)}(a), z_{-(n-2)-(n-1)}^{(n-1)}(b)] \oplus [z_{n-2}^{(n-1)}(a), z_{-(n-2)-(n-1)}^{(n-1)}(-b)] \\
&= z_{-(n-1)}^{(n-1)}(-ab)z_{-(n-2)-2(n-1)}^{(n-1)}(ab^2) \oplus z_{-(n-1)}^{(n-1)}(ab)z_{-(n-2)-2(n-1)}^{(n-1)}(ab^2) \\
&= \left(z_{-(n-1)}^{(n-1)}(-ab) \oplus z_{-(n-1)}^{(n-1)}(ab)\right)\left(z_{-(n-2)-2(n-1)}^{(n-1)}(ab^2) \oplus z_{-(n-2)-2(n-1)}^{(n-1)}(ab^2)\right) \\
&= z_{-(n-1)}^{(n)}(-ab)z_{-(n-2)-2(n-1)}^{(n)}(ab^2).
\end{aligned}
$$

Hence all the relations hold and thus we have a representation of some central extension of $\mathrm{O}^{\circ}_{2n+1}(q)$.

We next determine the diagonal elements of this representation, which follow directly from the corresponding proof in Theorem 5.2.5:

$$\begin{aligned}
h_1^{(n)}(b) &= I_{2^{n-2}} \oplus b^{-1} I_{2^{n-2}} \oplus b I_{2^{n-2}} \oplus I_{2^{n-2}} \\
h_i^{(n)}(b) &= h_{i-1}^{(n-1)}(b) \oplus h_{i-1}^{(n-1)}(b) && \text{for } 2 \leq r \leq n-1 \\
h_n^{(n)}(b) &= h_{n-1}^{(n-1)}(b) \oplus h_{n-1}^{(n-1)}(b).
\end{aligned}$$

It follows by induction that the matrix

$$h_n^{(n)}(b) = z_n^{(n)}(b) z_{-n}^{(n)}(-b^{-1}) z_n^{(n)}(b) z_n^{(n)}(-1) z_{-n}^{(n)}(1) z_n^{(n)}(-1)$$

is diagonal and of the form $\mathrm{diag}(b^{-1}, b, \ldots, b^{-1}, b)$. Hence when q is odd we can take $b = -1$ to show that $-I_{2^n}$ is in the group generated by the above generators, and hence this group has a centre of size at least 2. Since the Schur multiplier of $\mathrm{O}_{2n+1}^{\circ}(q)$ is $(2, q-1)$ from [19, Table 2.2], we have a representation of $\mathrm{Spin}_{2n+1}^{\circ}(q)$. The proof that this module is irreducible, and hence is the spin representation, is similar to the proof of irreducibility as given in Theorem 5.2.5. □

Lemma 5.3.6. *[33, Proposition 5.4.9] Let $G = \mathrm{Spin}_{2n+1}^{\circ}(q)$. Then G embeds into a classical group Ω as described below:*

(i) If q is even, then $\Omega = \Omega_{2^n}^{+}(q)$.

(ii) If $n \equiv 0, 3 \mod 4$ and q is odd, then $\Omega = \Omega_{2^n}^{+}(q)$.

(iii) If $n \equiv 1, 2 \mod 4$ and q is odd, then $\Omega = \mathrm{Sp}_{2^n}(q)$.

Lemma 5.3.7. *Let $G = \mathrm{Spin}_{2n+1}^{\circ}(q)$ denote the spin representation, generated by the generators as given above. Then G preserves the form $f^{(n)}$ defined as follows:*

$$\begin{aligned}
f^{(1)} &= \mathrm{antidiag}(-1, 1) \\
f^{(2)} &= \mathrm{antidiag}(-1, -1, 1, 1) \\
f^{(n+2)} &= -f^{(n)} \mathbin{\widehat{\oplus}} -f^{(n)} \mathbin{\widehat{\oplus}} f^{(n)} \mathbin{\widehat{\oplus}} f^{(n)}.
\end{aligned}$$

Proof. A direct computation, or via Lemma 5.2.10. □

5.3.3 Automorphisms

We will proceed using similar techniques to those used in Section 5.2.4.

Using [47, Section 2.2] we obtain generators for $\Omega^{\circ}_{2n+1}(q)$ in its natural representation:

$$\begin{aligned}
x_i^{(n)}(t) &= I_{2n+1} + t(E_{i+1,i} - E_{2n+2-i,2n+1-i}) && \text{for } 1 \le i \le n-1,\\
x_{-i}^{(n)}(t) &= x_i^{(n)}(t)^T,\\
x_n^{(n)}(t) &= I_{2n+1} + t(2E_{n+2,n+1} - E_{n+1,n} - E_{n+2,n}),\\
x_{-n}^{(n)}(t) &= I_{2n+1} + t(2E_{n,n+1} - E_{n+1,n+2} - E_{n,n+2}).
\end{aligned}$$

Note that the generator $x_n^{(n)}(t)$ is derived from the corresponding element of the Lie algebra given by $t(2E_{n+2,n+1} - E_{n+1,n})$.

Let χ be an $\mathbb{F}_q$-character of P, and suppose $\chi(\alpha_i) = \lambda_i$. Recall the notation of Section 5.1.4. If $h(\chi)$ is induced by a diagonal matrix $\mathrm{diag}(a_1, \ldots, a_{2n})$, we obtain the following conditions on the a_i from the above generators:

$$\begin{aligned}
\lambda_i &= a_{i+1}^{-1}a_i = a_{2n+2-i}^{-1}a_{2n+1-i} && \text{for } 1 \le i \le n-1,\\
\lambda_n &= a_{n+2}^{-1}a_{n+1} = a_{n+1}^{-1}a_n,\\
\lambda_n^2 &= a_{n+2}^{-1}a_n.
\end{aligned}$$

Similarly to before, we solve this by setting $a_{n+1} = t$ and obtaining:

$$\begin{aligned}
a_1 &= t\lambda_1\lambda_2\ldots\lambda_n, & a_{n+1} &= t,\\
a_2 &= t\lambda_2\lambda_3\ldots\lambda_n, & a_{n+2} &= t\lambda_n^{-1},\\
a_3 &= t\lambda_3\lambda_4\ldots\lambda_n, & a_{n+3} &= t\lambda_{n-1}^{-1}\lambda_n^{-1},\\
&\vdots & a_{n+4} &= t\lambda_{n-2}^{-1}\lambda_{n-2}^{-1}\lambda_n^{-1},\\
a_{n-1} &= t\lambda_{n-1}\lambda_n, & &\vdots\\
a_n &= t\lambda_n, & a_{2n+1} &= t\lambda_1^{-1}\lambda_2^{-1}\ldots\lambda_n^{-1}.
\end{aligned}$$

A generator for the diagonal outer automorphism δ of $\Omega^{\circ}_{2n+1}(q)$ is given by $\mathrm{diag}(\nu, 1, \ldots, 1, \nu^{-1})$ for ν a primitive element of $\mathbb{F}_q^*$, which corresponds to the character $(\nu, 1, 1, \ldots, 1)$. As before let $d_\delta^{(n)}$ induce δ on the spin representation $\mathrm{Spin}^{\circ}_{2n+1}(q)$.

Lemma 5.3.8. $d_\delta^{(n)} = \mathrm{diag}(1, \ldots, 1, \nu^{-1}, \ldots, \nu^{-1})$.

Proof. It is a straightforward computation to check that conjugation by the matrix $\mathrm{diag}(1, \ldots, 1, \nu^{-1}, \ldots, \nu^{-1})$ sends $z_1^{(n)}(b)$ to $z_1^{(n)}(\nu b)$. For the remaining generators, note that $\mathrm{diag}(1, \ldots, 1, \nu^{-1}, \ldots, \nu^{-1})$ is a direct sum of two 2^{n-1}-dimensional scalar matrices, and all other generators $z_i^{(n)}(b)$ are direct sums of two 2^{n-1}-dimensional matrices; hence $\mathrm{diag}(1,\dots,1,\nu^{-1},\dots,\nu^{-1})$ will commute with all the remaining generators. Hence conjugation by $\mathrm{diag}(1,\dots,1,\nu^{-1},\dots,\nu^{-1})$ induces the diagonal automorphism corresponding to the character $(\nu,1,\dots,1)$, so that we can take $d_\delta^{(n)} = \mathrm{diag}(1,\dots,1,\nu^{-1},\dots,\nu^{-1})$. $\square$

5.3.4 Results

Lemma 5.3.9. *Let $\Omega = \mathrm{Sp}_{2^n}(q)$ with $n \equiv 1,2 \mod 4$ and $q = p^e$ with $p \neq 2$, let $G = 2{}^{\cdot}\Omega^\circ_{2n+1}(q)$ be an $\mathcal{S}_2^*$-candidate subgroup of Ω, with G the action group of the spin representation, and let $S = N_\Omega(G)$. Then we have $S = G$. We have a single Ω-class of subgroups isomorphic to S, with class stabiliser in $\mathrm{CSp}_{2^n}(q)$ given by $\langle\delta_\Omega\rangle$ with δ_Ω inducing δ_G. We also have that ϕ_Ω induces ϕ_G.*

Proof. The shape of Ω is from Lemma 5.3.6, and the structure of the preserved form is from Lemma 5.3.7; in particular the form preserved by G is antidiagonal. From Lemma 5.3.8, it is straightforward to check that the matrix $d_\delta^{(n)}$ inducing δ on G scales the form by ν^{-1}, a primitive element of $\mathbb{F}_q^*$. Thus $d_\delta^{(n)}$ lies in $\mathrm{CSp}_{2^n}(q) \setminus \mathrm{Sp}_{2^n}(q)$ and so the automorphism δ_G of G is induced by δ_Ω. Results on the field automorphisms follow from Corollary 4.4.2. $\square$

Lemma 5.3.10. *Let $\Omega = \Omega^+_{2^n}(q)$ with $n \equiv 0,3 \mod 4$, $n > 3$ and $q = p^e$ with $p \neq 2$, let $G = 2{}^{\cdot}\Omega^\circ_{2n+1}(q)$ be an $\mathcal{S}_2^*$-candidate subgroup of Ω, with G the action group of the spin representation, and let $S = N_\Omega(G)$. Then we have $S = G$. We have four Ω-classes of subgroups isomorphic to S, with class stabiliser in $\mathrm{CGO}^+_{2^n}(q)$ given by $\langle\delta_\Omega\rangle$, with δ_Ω inducing δ_G. We also have that ϕ_Ω induces ϕ_G.*

Proof. Similarly to Lemma 5.3.9, we can determine the form preserved by G from Lemmas 5.3.6 and 5.3.7, and it also follows that $d_\delta^{(n)}$ lies in the conformal group of Ω. Hence, similarly to Lemma 5.2.15, this will be induced by δ_Ω or $\delta_\Omega\gamma_\Omega$ depending on whether the spinor norm of $\nu(d_\delta^{(n)})^2$ is 1 or -1 respectively. This matrix is $\mathrm{diag}(\nu,\dots,\nu,\nu^{-1},\dots,\nu^{-1})$, which is a product of 2^{n-1} reflections, each of the form $I_{2^n} + \nu E_{i,i} - \nu^{-1}E_{2^n+1-i,2^n+1-i}$ for $1 \leq i \leq 2^{n-1}$. Since $n > 3$ this has spinor norm 1, so that δ_G is induced by δ_Ω. Results on the field automorphisms follow from Corollary 4.4.2. $\square$

Lemma 5.3.11. *Let $\Omega = \Omega^+_{2^n}(q)$ with $n > 2$ and $q = 2^e$, let $G = \Omega^\circ_{2n+1}(q) \cong \mathrm{Sp}_{2n}(q)$ be an $\mathcal{S}_2^*$-candidate subgroup of Ω, with G the action group of the spin representation, and let $S = N_\Omega(G)$. Then we have $S = G$. We have a single Ω-class of subgroups* *isomorphic to S, with trivial class stabiliser in* $\mathrm{CGO}_{2n}^{+}(q)$. *We also have that* ϕ_Ω *induces* ϕ_G.

Proof. The form preserved is due to Lemma 5.3.6. The only outer automorphisms of Ω are field automorphisms, and so the result follows from Corollary 4.4.2. $\square$

5.4 Constructing the half-spin representation $\mathrm{HSpin}_{2n}^{-}(q)$

5.4.1 Generators

The generators $z_i^{(n)}(t)$ of the half-spin representation $\mathrm{HSpin}_{2n}^{-}(q)$ can be found from the generators $y_i(b)$ of $\mathrm{HSpin}_{2n}^{+}(q^2)$ as given in Section 5.2.2 (see Example 5.1.9):

$$\begin{aligned} z_i^{(n)}(b) &= y_i^{(n)}(b) && \forall\, i \le n-2, \forall\, b \in \mathbb{F}_q^* \\ z_{n-1}^{(n)}(r) &= y_{n-1}^{(n)}(r) y_n^{(n)}(\bar{r}) && \forall\, r \in \mathbb{F}_{q^2}^* \end{aligned}$$

where $\bar{r} = r^q$ denotes the image of $r \in \mathbb{F}_{q^2}^*$ under the field automorphism σ of order 2. Again the full set of generators consist of the $z_i^{(n)}(t)$ and their transposes.

Remark 5.4.1. (1) Note that matrices corresponding to short roots (in our case, the roots α_n and $\alpha_{n-1} + \alpha_n$) take their argument from $\mathbb{F}_{q^2}^*$, whereas matrices corresponding to long roots take their argument from $\mathbb{F}_q^*$. To distinguish between the two, we will use r and s to denote elements of $\mathbb{F}_{q^2}^*$, and a and b to denote elements of $\mathbb{F}_{q^2}^*$. We will use t as the argument when considering both long and short roots together.

(2) Notice that if we take the last generator $z_{n-1}^{(n)}(r)$ to only take arguments in $\mathbb{F}_q^*$ rather than $\mathbb{F}_{q^2}^*$, we would have the generators of $\mathrm{Spin}_{2n-1}^{\circ}(q)$ as given in Section 5.3.1. In particular, by considering the restriction of the above generators to the generators of $\mathrm{Spin}_{2n-1}^{\circ}(q)$, we see that the structure constants $\epsilon_{\alpha,\beta}$ and $\eta_{\alpha,\beta}$ must agree with the structure constants found in Examples 5.1.3 and 5.1.5, and we can use these and Theorem 5.1.7 to write down the relations that these generators must satisfy in order to generate a representation of the half-spin representation $\mathrm{HSpin}_{2n}^{-}(q)$.

5.4.2 Proof of correctness of the generators

Lemma 5.4.2. *Assume the relations in Theorem 5.1.7 hold. Then we have formulae for $z_{i+2j}(b)$ and $z_{i+j}(r)$:*

$$z_{i+2j}^{(n)}(a) = z_{-j}^{(n)}(1)z_j^{(n)}(-1)z_i^{(n)}(-a)z_j^{(n)}(1)z_{-j}^{(n)}(-1)$$
$$z_{i+j}^{(n)}(r) = \left[z_i^{(n)}(1), z_j^{(n)}(r)\right] z_{i+2j}^{(n)}(r\bar{r}).$$

Proof. Similar to Lemma 5.3.1. □

Remark 5.4.3. (1) As in Remark 5.3.2, we can also obtain the alternate formula:

$$z_{i+2j}^{(n)}(a) = z_{-j}^{(n)}(-1)z_j^{(n)}(1)z_i^{(n)}(-a)z_j^{(n)}(-1)z_{-j}^{(n)}(1).$$

(2) Note that the root $\alpha_i + 2\alpha_j$ is a long root, so the expression $z_{i+2j}^{(n)}(a)$ is only valid for $a \in \mathbb{F}_q^*$. The root $\alpha_i + \alpha_j$ is short, so $z_{i+j}^{(n)}(r)$ is valid for $r \in \mathbb{F}_{q^2}^*$.

(3) Similarly to before, for the half-spin representation we can compute that $z_{-i-j}^{(n)}(r) = z_{i+j}^{(n)}(\bar{r})^T$ and $z_{-i-2j}^{(n)}(a) = z_{i+2j}^{(n)}(a)^T$.

Lemma 5.4.4. *Let $z_i^{(n)}(t)$ be as defined in Section 5.4.1. Then if the relations in Theorem 5.1.7 hold, the following formulae hold, for $n > 2$:*

(1) $z_{(1)+(2)}^{(2)}(r) = I_4 - rE_{3,1} - \bar{r}E_{4,2}$.

(2) $z_{(1)+2(2)}^{(2)}(a) = I_4 - aE_{4,1}$.

(3) $z_i^{(n)}(a) = z_{i-1}^{(n-1)}(a) \oplus z_{i-1}^{(n-1)}(a)$ *for all i with $2 \le i \le n-2$.*

(4) $z_{n-1}^{(n)}(r) = z_{n-2}^{(n-1)}(r) \oplus z_{n-2}^{(n-1)}(-\bar{r})$.

(5) $z_{-(n-1)}^{(n)}(r) = z_{-(n-2)}^{(n-1)}(r) \oplus z_{-(n-2)}^{(n-1)}(-\bar{r})$.

(6) $z_{(n-2)+(n-1)}^{(n)}(r) = z_{(n-3)+(n-2)}^{(n-1)}(r) \oplus z_{(n-3)+(n-2)}^{(n-1)}(-\bar{r})$.

(7) $z_{(n-2)+2(n-1)}^{(n)}(a) = z_{(n-3)+2(n-2)}^{(n-1)}(a) \oplus z_{(n-3)+2(n-2)}^{(n-1)}(a)$.

Proof. Similar to Lemma 5.3.3.

□

Theorem 5.4.5. *Let $n \ge 3$ and $q = p^e$ be a prime power. Let B denote a basis of $\mathbb{F}_q$ viewed as a $\mathbb{F}_p$-vector space, and extend B to a basis R of $\mathbb{F}_{q^2}$, again viewed as a $\mathbb{F}_p$-vector space. For $r \in \mathbb{F}_{q^2}$, let $\bar{r} = r^q$, the field automorphism which fixes $\mathbb{F}_q$.*

Let $z_i^{(n)}(t)$ be as given in Section 5.4.1. Then the group generated by $\{z_i^{(n)}(b) : b \in B, i = 1, \dots, n-2\}$ and $\{z_{n-1}^{(n)}(r) : r \in R\}$, plus the matrices for the corresponding negative roots $\{z_i^{(n)}(b)^T : b \in B, i = 1, \dots, n-2\}$ and $\{z_{n-1}^{(n)}(\bar{r})^T : r \in R\}$, is the half-spin representation $\mathrm{HSpin}_{2n}^{-}(q)$.

Proof. We check that these generators satisfy all the relations of Theorem 5.1.7.

From Remark 5.4.1, we see that any relations not involving $z_{n-1}^{(n)}(r)$ or its conjugate transpose are precisely the relations checked in Theorem 5.3.5, and in turn in Theorem 5.2.5. Thus it suffices to check only the relations involving $z_{n-1}^{(n)}(r)$ and its conjugate transpose.

Since $y_{n-1}^{(n)}(r)$ and $y_n^{(n)}(\bar{r})$ both commute with $y_i^{(n)}(b)$ for $i < n-2$ by Theorem 5.2.5, it follows immediately that $z_{n-1}^{(n)}(r)$ commutes with $z_i^{(n)}(b)$ for $i < n-2$, and thus we only need to check the relations involving roots in the twisted Dynkin diagram $\mathfrak{B}_{n-1}$ with support in $\{n-2, n-1\}$.

The relations that we need to check are due to Theorem 5.1.7, and are listed explicitly below, where $i = n-2$ and $j = n-1$:

$$
\begin{array}{ll}
[z_i^{(n)}(a), z_j^{(n)}(r)] = z_{i+j}^{(n)}(ar) z_{i+2j}^{(n)}(-ar\bar{r}) & [z_{-i}^{(n)}(a), z_{-j}^{(n)}(r)] = z_{-i-j}^{(n)}(-ar) z_{-i-2j}^{(n)}(-ar\bar{r}) \\
[z_i^{(n)}(a), z_{i+j}^{(n)}(r)] = 1 & [z_{-i-j}^{(n)}(a), z_{-i}^{(n)}(r)] = 1 \\
[z_i^{(n)}(a), z_{i+2j}^{(n)}(b)] = 1 & [z_{-i}^{(n)}(a), z_{-i-2j}^{(n)}(b)] = 1 \\
[z_i^{(n)}(a), z_{-j}^{(n)}(r)] = 1 & [z_j^{(n)}(a), z_{-i}^{(n)}(r)] = 1 \\
[z_i^{(n)}(a), z_{-i-j}^{(n)}(r)] = z_{-j}^{(n)}(-ar) z_{-i-2j}^{(n)}(ar\bar{r}) & [z_{-i}^{(n)}(a), z_{i+j}^{(n)}(r)] = z_j^{(n)}(ar) z_{i+2j}^{(n)}(ar\bar{r}) \\
[z_i^{(n)}(a), z_{-i-2j}^{(n)}(b)] = 1 & [z_{i+2j}^{(n)}(a), z_{-i}^{(n)}(b)] = 1 \\
[z_j^{(n)}(r), z_{i+j}^{(n)}(s)] = z_{i+2j}^{(n)}(r\bar{s} + \bar{r}s) & [z_{-i-j}^{(n)}(r), z_{-j}^{(n)}(s)] = z_{-i-2j}^{(n)}(r\bar{s} + \bar{r}s) \\
[z_j^{(n)}(r), z_{i+2j}^{(n)}(a)] = 1 & [z_{-i-2j}^{(n)}(a), z_{-j}^{(n)}(r)] = 1 \\
[z_j^{(n)}(r), z_{-i-j}^{(n)}(s)] = z_{-i}^{(n)}(r\bar{s} + \bar{r}s) & [z_{i+j}^{(n)}(r), z_{-j}^{(n)}(s)] = z_i^{(n)}(r\bar{s} + \bar{r}s) \\
[z_{-i-2j}^{(n)}(a), z_j^{(n)}(r)] = z_{-i-j}^{(n)}(ar) z_{-i}^{(n)}(-ar\bar{r}) & [z_{i+2j}^{(n)}(a), z_{-j}^{(n)}(r)] = z_{i+j}^{(n)}(-ar) z_i^{(n)}(-ar\bar{r}) \\
[z_{i+j}^{(n)}(r), z_{i+2j}^{(n)}(a)] = 1 & [z_{-i-2j}^{(n)}(a), z_{-i-j}^{(n)}(r)] = 1 \\
[z_{-i-2j}^{(n)}(a), z_{i+j}^{(n)}(r)] = z_{-j}^{(n)}(-ar) z_i^{(n)}(ar\bar{r}) & [z_{i+2j}^{(n)}(a), z_{-i-j}^{(n)}(r)] = z_j^{(n)}(ar) z_{-i}^{(n)}(ar\bar{r})
\end{array}
$$

As in Theorem 5.2.5 and Theorem 5.3.5, we check these results for the base case $n = 3$ by hand or by computer. For higher dimensions, the results again follow by straightforward induction using Lemma 5.4.4, and we only supply two sample calculations; the remaining ones are similar.

$$\begin{aligned}
&\left[z^{(n)}_{(n-2)+(n-1)}(r), z^{(n)}_{-(n-1)}(s)\right]\\
&= \left[z^{(n-1)}_{(n-3)+(n-2)}(r), z^{(n-1)}_{-(n-2)}(s)\right] \oplus \left[z^{(n-1)}_{(n-3)+(n-2)}(-\bar{r}), z^{(n-1)}_{-(n-2)}(-\bar{s})\right]\\
&= z^{(n-1)}_{n-3}(r\bar{s}+\bar{r}s) \oplus z^{(n-1)}_{n-3}(\bar{r}s+r\bar{s})\\
&= z^{(n)}_{n-2}(r\bar{s}+\bar{r}s).
\end{aligned}$$

$$\begin{aligned}
&\left[z^{(n)}_{-(n-2)-2(n-1)}(a), z^{(n)}_{i+j}(r)\right]\\
&= \left[z^{(n-1)}_{-(i-3)-2(i-2)}(a), z^{(n-1)}_{(n-3)+(n-2)}(r)\right] \oplus \left[z^{(n-1)}_{-(n-3)-2(n-2)}(a), z^{(n-1)}_{(n-3)+(n-2)}(-\bar{r})\right]\\
&= \left(z^{(n-1)}_{-(n-2)}(-ar)z^{(n-1)}_{n-3}(ar\bar{r})\right) \oplus \left(z^{(n-1)}_{-(n-2)}(a\bar{r})z^{(n-1)}_{n-3}(ar\bar{r})\right)\\
&= \left(z^{(n-1)}_{-(n-2)}(-ar) \oplus z^{(n-1)}_{-(n-2)}(a\bar{r})\right)\left(z^{(n-1)}_{n-3}(ar\bar{r}) \oplus z^{(n-1)}_{n-3}(ar\bar{r})\right)\\
&= z^{(n)}_{-(n-1)}(-ar)z^{(n)}_{n-2}(ar\bar{r})
\end{aligned}$$

where the final equality requires the fact that $a \in \mathbb{F}_q$ and hence $\bar{a} = a$.

We can determine which scalars lie in this group from the computation of the scalars in the half-spin representation $\mathrm{HSpin}^+_{2n}(q^2)$. When q is even the group has trivial centre. Otherwise note that, using the notation of Theorem 5.2.5, the group of diagonal elements of $\mathrm{HSpin}^-_{2n}(q)$ is generated by $h^{(n)}_i(t)$ for $1 \le i \le n-2$ and $h^{(n)}_{n-1}(r)h^{(n)}_n(r^q)$. In particular, the matrix $-I_{2^{n-1}}$ is always contained in this representation. From the shape of the Schur multiplier and the fact that the module we have constructed is absolutely irreducible, it follows that this is the entire centre when n is odd or $q \equiv 1 \mod 4$. When $q \equiv 3 \mod 4$, note that $q^2 \equiv 1 \mod 4$ and so the half-spin representation $\mathrm{HSpin}^+_{2n}(q^2)$ has a cyclic centre of order 4, generated by $\nu^{\frac{q^2-1}{4}} I_{2^{n-1}}$. Since $q \equiv 3 \mod 4$, $\nu^{\frac{q^2-1}{4}}$ does not lie in $\mathbb{F}_q$, so that the element $h^{(n)}_{n-1}\left(\nu^{\frac{q^2-1}{4}}\right) h^{(n)}_n\left(\nu^{\frac{q^2-1}{4}}\right)$ lies inside this representation of $\mathrm{HSpin}^-_{2n}(q)$, and hence so does $\nu^{\frac{q^2-1}{4}} I_{2^{n-1}}$ as constructed in the proof of Theorem 5.2.5.

Absolute irreducibility again follows from similar arguments to those for Theorem 5.2.5 and Theorem 5.3.5, and we can also check the central elements in a similar manner to Theorem 5.3.5 to ensure that this is the half-spin representation, again making use of [33, Proposition 5.4.11] to confirm that the only absolutely irreducible representation of dimension 2^{n-1} of a central extension of $\mathrm{O}^-_{2n}(q)$ is the half-spin representation. □

Remark 5.4.6. As in the case for $\mathrm{HSpin}^+_{2n}(q)$, the above generators describe one of the half-spin representations. We obtain the other half-spin representation by defining $z_{n-1}^{(n)}(r)$ to be $y_n^{(n)}(r)y_{n-1}^{(n)}(\bar{r})$ instead, which is again obtained by applying the graph automorphism γ to the generators $y_i^{(n)}(t)$.

Lemma 5.4.7. *[33, Proposition 5.4.9] Let $G = \mathrm{HSpin}_{2n}^{-}(q)$. Then the half-spin representation embeds G into a classical group Ω as described below.*

(i) If n is odd, then $\Omega = \mathrm{SU}_{2^{n-1}}(q)$.

(ii) If q is even and n is even, then $\Omega = \Omega_{2^{n-1}}^{+}(q^2)$.

(iii) If q is odd and $n \equiv 0 \mod 4$, then $\Omega = \Omega_{2^{n-1}}^{+}(q^2)$.

(iv) If q is odd and $n \equiv 2 \mod 4$, then $\Omega = \mathrm{Sp}_{2^{n-1}}(q^2)$.

Remark 5.4.8. Recall that the half-spin representation $\mathrm{HSpin}_{2n}^{-}(q)$ embeds inside the half-spin representation $\mathrm{HSpin}_{2n}^{+}(q^2)$. In particular when n is even we can use Lemmas 5.2.9 and 5.2.10 to determine the form preserved by the half-spin representation in this case.

Recall the definition of $\widehat{\oplus}$ as given in Definition 1.6.3.

Lemma 5.4.9. *Let $G = \mathrm{HSpin}_{2n}^{-}(q)$, with n odd. Then the half-spin representation of G preserves the unitary form f_n as described below:*

$$\begin{aligned} f_3 &= \mathrm{antidiag}(1, 1, -1, -1) \\ f_{n+2} &= f_n \mathbin{\widehat{\oplus}} f_n \mathbin{\widehat{\oplus}} -f_n \mathbin{\widehat{\oplus}} -f_n. \end{aligned}$$

Proof. We proceed by induction. The case $n = 3$ is a direct computation.

Suppose the result holds for f_n. We first check that $z_{n+1}^{(n+2)}(r)$ preserves f_n, as this is the hardest case to consider.

For ease of notation, we define $f_{n+1} = f_n \mathbin{\widehat{\oplus}} f_n$ for n odd, so that $f_{n+2} = f_{n+1} \mathbin{\widehat{\oplus}} -f_{n+1}$. Then we have via Lemma 5.4.4 and Lemma 1.6.5 that:

$$\begin{aligned} &z_{n+1}^{(n+2)}(r)\ f_{n+2}\ z_{-n-1}^{(n+2)}(\bar{r}) \\ &= \left(z_n^{(n+1)}(r) \oplus z_n^{(n+1)}(-\bar{r})\right) \left(f_{n+1} \mathbin{\widehat{\oplus}} - f_{n+1}\right) \left(z_{-n}^{(n+1)}(\bar{r}) \oplus z_{-n}^{(n+1)}(-r)\right) \\ &= \left(z_n^{(n+1)}(-\bar{r})\ f_{n+1} \mathbin{\widehat{\oplus}} - z_n^{(n+1)}(r)\ f_{n+1}\right) \left(z_{-n}^{(n+1)}(\bar{r}) \oplus z_{-n}^{(n+1)}(-r)\right) \\ &= \left(z_n^{(n+1)}(-\bar{r})\ f_{n+1}\ z_{-n}^{(n+1)}(\bar{r})\right) \mathbin{\widehat{\oplus}} \left(-z_n^{(n+1)}(r)\ f_{n+1}\ z_{-n}^{(n+1)}(-r)\right). \end{aligned}$$

We first consider the left-hand term in the anti-direct sum above:

$$
\begin{aligned}
&\left(z_n^{(n+1)}(-\bar{r})\ f_{n+1}\ z_{-n}^{(n+1)}(\bar{r})\right)\\
&= \left(z_{n-1}^{(n)}(-\bar{r}) \oplus z_{n-1}^{(n)}(r)\right)\left(f_n \mathbin{\widehat{\oplus}} f_n\right)\left(z_{-n+1}^{(n)}(\bar{r}) \oplus z_{-n+1}^{(n)}(-r)\right)\\
&= \left(z_{n-1}^{(n)}(r)\ f_n \mathbin{\widehat{\oplus}} z_{n-1}^{(n)}(-\bar{r})\ f_n\right)\left(z_{-n+1}^{(n)}(\bar{r}) \oplus z_{-n+1}^{(n)}(-r)\right)\\
&= \left(z_{n-1}^{(n)}(r)\ f_n\ z_{-n+1}^{(n)}(\bar{r})\right) \mathbin{\widehat{\oplus}} \left(z_{n-1}^{(n)}(-\bar{r})\ f_n\ z_{-n+1}^{(n)}(-r)\right)\\
&= f_n \mathbin{\widehat{\oplus}} f_n = f_{n+1},
\end{aligned}
$$

where the last line follows by induction. Similarly for the right-hand term we have:

$$
\begin{aligned}
&-\left(z_n^{(n+1)}(r)\ f_{n+1}\ z_{-n}^{(n+1)}(-r)\right)\\
&= -\left(z_{n-1}^{(n)}(r) \oplus z_{n-1}^{(n)}(-\bar{r})\right)\left(f_n \mathbin{\widehat{\oplus}} f_n\right)\left(z_{-n+1}^{(n)}(-r) \oplus z_{-n+1}^{(n)}(\bar{r})\right)\\
&= -\left(z_{n-1}^{(n)}(-\bar{r})\ f_n \mathbin{\widehat{\oplus}} z_{n-1}^{(n)}(r)\ f_n\right)\left(z_{-n+1}^{(n)}(-r) \oplus z_{-n+1}^{(n)}(\bar{r})\right)\\
&= -\left(z_{n-1}^{(n)}(-\bar{r})\ f_n\ z_{-n+1}^{(n)}(-r)\right) \mathbin{\widehat{\oplus}} \left(z_{n-1}^{(n)}(r)\ f_n\ z_{-n+1}^{(n)}(\bar{r})\right)\\
&= -\left(f_n \mathbin{\widehat{\oplus}} f_n\right) = -f_{n+1}.
\end{aligned}
$$

Hence we have

$$
z_{n+1}^{(n+2)}(r)\ f_{n+2}\ z_{-n-1}^{(n+2)}(\bar{r}) = f_{n+1} \mathbin{\widehat{\oplus}} -f_{n+1} = f_{n+2},
$$

so that $z_{n+1}^{(n+2)}(r)$ also preserves the unitary form f_{n+2}.

The remaining generators are simpler, since all their entries lie in $\mathbb{F}_q$ and hence are fixed by the field automorphism. A similar inductive argument holds for $z_i^{(n)}(a)$ with $3 \leq i \leq n-2$, and a direct computation for $z_1^{(n)}(a)$ and $z_2^{(n)}(a)$ confirms that these generators also preserve f_n.

□

5.4.3 Automorphisms

To determine which diagonal automorphisms of the action groups of the spin representations are outer as elements of the ambient classical group Ω, we will make use of Theorem 5.1.14. Throughout this section we will assume that q is odd; there are no non-trivial diagonal automorphisms when q is even. We will also use r^q in place of $\bar{r}$ for $r \in \mathbb{F}_q$.

Note that in $\mathfrak{D}_n$, a character $(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \in \mathbb{F}_{q^2}$ is self-conjugate if and only if $\lambda_i \in \mathbb{F}_q$ for $i \leq n-2$ and $\lambda_{n-1} = \lambda_n^q$. Let $A = (A_{i,j})$ denote the $n \times n$ Cartan matrix of $\mathfrak{D}_n$. Then we have that $(\lambda_1, \ldots, \lambda_n)$ is a $\mathbb{F}_{q^2}$-character of P which extends to a $\mathbb{F}_{q^2}$-character $(\mu_1, \ldots, \mu_n)$ of Q if and only if $\lambda_i = \prod_{j=1}^{n} \mu_i^{A_{i,j}}$ for all i. In particular, if $B = (B_{i,j})$ denotes the inverse of the Cartan matrix A, then we must also have $\mu_j = \prod_{i=1}^{n} \lambda_i^{B_{i,j}}$ for all j. Note that although $A_{i,j} \in \mathbb{Z}$, we generally will only have $B_{i,j} \in \mathbb{Q}$; in particular, if $B_{i,j}$ has denominator k, the definition of $\lambda_i^{B_{i,j}}$ may depend on a choice of k-th root of λ_i.

Firstly, consider the self-conjugate $\mathbb{F}_{q^2}$-character of P given by the character $\lambda = (1, \ldots, 1, \nu, \nu^q)$, where ν is a primitive element of $\mathbb{F}_{q^2}^*$. Then if there exists a $\mathbb{F}_{q^2}$-character of Q given by $\mu = (\mu_1, \ldots, \mu_n)$ which restricts to λ, then in particular we must have $\mu_1 = \prod_{i=1}^{n} \lambda_i^{B_{i,1}} = \nu^{\frac{q+1}{2}}$, so μ_1 must be a square root of ν^{q+1}. However, ν^{q+1} has multiplicative order $q-1$, hence it is a primitive element of $\mathbb{F}_q^*$, and thus has no roots in $\mathbb{F}_q$, which means that no self-conjugate $\mathbb{F}_{q^2}$-character of Q exists which restricts to λ. Hence the character λ induces an outer automorphism of $\Omega_{2n}^-(q)$. In fact, λ induces the automorphism δ in all cases. This is clear when $4|n$ or when $q \equiv 1 \mod 4$, since there is only one nontrivial outer diagonal automorphism; when $n \equiv 2 \mod 4$ and $q \equiv 3 \mod 4$, we will show in the next lemma that the automorphism induced by $\lambda' = (1, \ldots, 1, \nu^2, \nu^{2q})$ is also an outer diagonal automorphism, and hence the automorphism induced by λ is (a conjugate of) δ.

Lemma 5.4.10. *Let $\lambda' = (1, \ldots, 1, \nu^2, \nu^{2q})$. Then λ' induces a nontrivial outer diagonal automorphism on $\Omega_{2n}^-(q)$ if and only if n is odd and $q \equiv 3 \mod 4$.*

Proof. Let $\lambda' = (1, \ldots, 1, \nu^2, \nu^{2q})$, and suppose $\mu = (\mu_1, \ldots, \mu_n)$ is a $\mathbb{F}_{q^2}$-character of Q which restricts to λ'. Then the following (derived from inverting the Cartan matrix) must hold, with ϵ_i and κ_i to be determined:

$$\begin{aligned}
\mu_1 &= \epsilon_1 \nu^{q+1}, \\
\mu_2 &= \epsilon_2 \nu^{2(q+1)}, \\
&\vdots \\
\mu_{n-2} &= \epsilon_{n-2} \nu^{(n-2)(q+1)}, \\
\mu_{n-1} &= \iota^{\kappa_{n-1}} \nu^{\frac{n+q(n-2)}{2}}, \\
\mu_n &= \iota^{\kappa_n} \nu^{\frac{n-2+qn}{2}},
\end{aligned}$$

where $\epsilon_i = 1$ if i is even, $\epsilon_i \in \{-1, 1\}$ if i is odd, ι is a fixed primitive fourth root of unity in $\mathbb{F}_{q^2}$, and $\kappa_{n-1}, \kappa_n \in \{0, 1, 2, 3\}$.

To determine the possibilities for these constants, we check the conditions $\lambda_i = \prod_{j=1}^{n} \mu_i^{A_{i,j}}$, which gives the following:

$$\begin{aligned}
1 = \lambda_1 &= \mu_1^2 \mu_2^{-1} = \epsilon_2, \\
1 = \lambda_2 &= \mu_1^{-1} \mu_2^2 \mu_3^{-1} = \epsilon_1 \epsilon_3, \\
&\vdots \\
1 = \lambda_i &= \mu_{i-1}^{-1} \mu_i^2 \mu_{i+1}^{-1} = \epsilon_{i-1} \epsilon_{i+1}, \\
&\vdots \\
1 = \lambda_{n-2} &= \mu_{n-3}^{-1} \mu_{n-2}^2 \mu_{n-1}^{-1} \mu_n^{-1} = \epsilon_{n-3} \iota^{-\kappa_{n-1}-\kappa_n}, \\
\nu^2 = \lambda_{n-1} &= \mu_{n-2}^{-1} \mu_{n-1}^2 = \nu^2 \epsilon_{n-2} \iota^{2\kappa_{n-1}}, \\
\nu^{2q} = \lambda_n &= \mu_{n-2}^{-1} \mu_n^2 = \nu^{2q} \epsilon_{n-2} \iota^{2\kappa_n}.
\end{aligned}$$

We also require μ to be a self-conjugate character, so we require $\mu_{n-1}^q = \mu_n$, and $\mu_i \in \mathbb{F}_q$ for $i \leq n-2$. Since ν^{q+1} has order $q-1$, this is a primitive element of $\mathbb{F}_q$, and so this latter condition is always satisfied. Checking the remaining conditions will depend on the parity of n, so we consider the cases separately.

Firstly, suppose that n is even. Then we have

$$\mu_{n-1}^q = \iota^{q\kappa_{n-1}} \nu^{\frac{nq+q^2(n-2)}{2}} = \iota^{q\kappa_{n-1}} \nu^{\frac{nq+(n-2)}{2}} = \iota^{q\kappa_{n-1}-\kappa_n} \mu_n$$

so that we require $q\kappa_{n-1} \equiv \kappa_n \mod 4$. From the conditions on $\lambda_{n-2}, \lambda_{n-1}$ and λ_n, we also require $\epsilon_1 = \iota^{-\kappa_{n-1}-\kappa_n}$ and $\kappa_{n-1}, \kappa_n \in \{0, 2\}$. Since q is odd, this means that we must have $\kappa_{n-1} = \kappa_n$, and hence $\epsilon_1 = 1$. We can take $\kappa_{n-1} = \kappa_n = 1$, and then λ' as a self-conjugate $\mathbb{F}_{q^2}$-character of P extends to μ, a self-conjugate $\mathbb{F}_{q^2}$-character of Q. Hence by Theorem 5.1.14 the automorphism induced by λ' is inner.

Now suppose that n is odd. Then we have

$$\begin{aligned}
\mu_{n-1}^q &= \iota^{q\kappa_{n-1}} \nu^{\frac{nq+q^2(n-2)}{2}} \\
&= \iota^{q\kappa_{n-1}} \nu^{\frac{q^2-1}{2}} \nu^{\frac{nq-1+q^2(n-1)}{2}} \\
&= -\iota^{q\kappa_{n-1}} \nu^{\frac{nq-1+(n-1)}{2}} \\
&= -\iota^{q\kappa_{n-1}-\kappa_n} \mu_n,
\end{aligned}$$

hence for μ to be self-conjugate we require $q\kappa_{n-1} \equiv \kappa_n + 2 \mod 4$. The conditions on $\lambda_{n-2}, \lambda_{n-1}$ and λ_n mean we also require $1 = \iota^{-\kappa_{n-1}-\kappa_n}$ and $\epsilon_1 = \iota^{2\kappa_{n-1}} = \iota^{2\kappa_n}$.

Hence we must have $\kappa_{n-1}, \kappa_n \in \{1,3\}$ and $\kappa_{n-1} + \kappa_n \equiv 0 \mod 4$. If $q \equiv 1 \mod 4$ then we can take $\kappa_{n-1} = 1$ and $\kappa_n = 3$ and the corresponding character μ is a $\mathbb{F}_{q^2}$-character of Q which restricts to λ' on P, hence λ' induces an inner automorphism by Theorem 5.1.14. If $q \equiv 3 \mod 4$ then for μ to be self-conjugate we require $\kappa_{n-1} \equiv \kappa_n \mod 4$ which is impossible given the other restrictions on the elements κ_i; hence in this case λ' induces a non-trivial outer diagonal automorphism, again by Theorem 5.1.14. □

Recall the generators of the half-spin representation in Section 5.4.1. Suppose we have a self-dual character $\lambda = (\lambda_1, \ldots, \lambda_{n-1}, \lambda_{n-1}^q)$. Then the automorphism $h(\lambda)$ acts on the generators as follows:

$$z_i^{(n)}(a) = y_i^{(n)}(a) \mapsto y_i^{(n)}(\lambda_i a) = z_i^{(n)}(\lambda_i a) \text{ for } i \leq n-2,$$
$$z_{n-1}^{(n)}(r) = y_{n-1}^{(n)}(r) y_n^{(n)}(r^q) \mapsto y_{n-1}^{(n)}(\lambda_{n-1} r) y_n^{(n)}(\lambda_{n-1}^q r^q) = z_{n-1}^{(n)}(\lambda_{n-1} r).$$

As before, let $d_\delta^{(n)}$ denote the diagonal matrix inducing the automorphism δ on the half-spin representation $\mathrm{HSpin}_{2n}^-(q)$.

Lemma 5.4.11.

(i) $d_\delta^{(3)} = \mathrm{diag}(1, \nu^{-1}, \nu^{-1}, \nu^{-q-1})$.

(ii) $d_\delta^{(n)} = d_\delta^{(n-1)} \oplus \nu^{-1} d_\delta^{(n-1)\sigma}$.

Proof. We will show that $d_\delta^{(n)}$ induces the automorphism $h(\lambda)$ for $\lambda = (1, \ldots, 1, \nu, \nu^q)$.

For $d_\delta^{(n)}$ to induce $h(\lambda)$ on the half-spin representation, it needs to commute with $z_i^{(n)}(a)$ for $i \leq n-2$, and conjugate $z_{n-1}^{(n)}(r)$ to $z_{n-1}^{(n)}(\nu r)$. We can check directly for $n = 3$ and $n = 4$ that the above relations hold. For larger n we proceed by induction. We have that

$$d_\delta^{(n)} = d_\delta^{(n-1)} \oplus \nu^{-1} d_\delta^{(n-1)\sigma} = d_\delta^{(n-2)} \oplus \nu^{-1} d_\delta^{(n-2)\sigma} \oplus \nu^{-1} d_\delta^{(n-2)\sigma} \oplus \nu^{-q-1} d_\delta^{(n-2)}$$

and so as in the proof of Theorem 5.2.11, $d_\delta^{(n)}$ commutes with $z_1^{(n)}(a)$. For i with $2 \leq i \leq n-2$, we assume inductively that $d_\delta^{(n-1)}$ commutes with $z_{i-1}^{(n-1)}(a)$. Then for $a \in \mathbb{F}_q^*$:

$$\begin{aligned}
d_\delta^{(n)} z_i^{(n)}(a) &= \left(d_\delta^{(n-1)} \oplus \nu^{-1} d_\delta^{(n-1)\sigma}\right)\left(z_{i-1}^{(n-1)}(a) \oplus z_{i-1}^{(n-1)}(a)\right) \\
&= \left(d_\delta^{(n-1)} z_{i-1}^{(n-1)}(a)\right) \oplus \left(\nu^{-1} d_\delta^{(n-1)\sigma} z_{i-1}^{(n-1)}(a)\right) \\
&= \left(d_\delta^{(n-1)} z_{i-1}^{(n-1)}(a)\right) \oplus \left(\nu^{-1}\left(d_\delta^{(n-1)} z_{i-1}^{(n-1)}(a)\right)^\sigma\right) \\
&= \left(z_{i-1}^{(n-1)}(a) d_\delta^{(n-1)}\right) \oplus \left(\nu^{-1}\left(z_{i-1}^{(n-1)}(a) d_\delta^{(n-1)}\right)^\sigma\right) \\
&= z_i^{(n)}(a) d_\delta^{(n)}
\end{aligned}$$

so that $d_\delta^{(n)}$ commutes with $z_i^{(n)}(a)$. Finally, it follows from:

$$\begin{aligned}
z_{n-1}^{(n)}(r) &= z_{n-2}^{(n-1)}(r) \oplus z_{n-2}^{(n-1)}(-r^q), \\
z_{n-2}^{(n-1)}(r)^{d_\delta^{(n-1)}} &= z_{n-2}^{(n-1)}(\nu r), \text{ and} \\
z_{n-2}^{(n-1)}(r)^{d_\delta^{(n-1)\sigma}} &= z_{n-2}^{(n-1)}(\nu^q r)
\end{aligned}$$

that $z_{n-1}^{(n)}(r)^{d_\delta^{(n)}} = z_{n-1}^{(n)}(\nu r)$.

□

Corollary 5.4.12. *The last entry of the diagonal of* $d_\delta^{(n)}$ *is* $\nu^{-q^{n-2}-q^{n-3}-\ldots-q-1} = \nu^{\frac{-q^{n-1}+1}{q-1}}$.

Proof. Immediate from Lemma 5.4.11. □

Lemma 5.4.13. $\det d_\delta^{(n)} = \nu^{a_n}$ *where* $a_n = \frac{-(q+1)^{n-1}+2^{n-1}}{q-1}$.

Proof. Let a_n be such that $\det d_\delta^{(n)} = \nu^{a_n}$. Then it follows from Lemma 5.4.11 that

$$\begin{aligned}
a_3 &= -q-3 \\
a_{n+1} &= f_n a_n + g_n
\end{aligned}$$

where $f_n = q+1$ and $g_n = -2^{n-1}$. Then we can use the formula for first-order non-homogeneous recurrence relations with variable coefficients (see for instance [55]), which gives us that:

$$
\begin{aligned}
a_n &= \left(\prod_{k=3}^{n-1} f_k\right)\left(a_3 + \sum_{m=3}^{n-1} \frac{g_m}{\prod_{k=3}^{m} f_k}\right)\\
&= (q+1)^{n-3}\left(-q-3-\sum_{m=3}^{n-1}\frac{2^{m-1}}{(q+1)^{m-2}}\right)\\
&= (q+1)^{n-3}\left(-q-3-\frac{4}{q+1}\left(\frac{1-\left(\frac{2}{q+1}\right)^{n-3}}{1-\left(\frac{2}{q+1}\right)}\right)\right)\\
&= (q+1)^{n-3}\left(-q-3-\frac{4}{q+1}\left(\frac{1-\left(\frac{2}{q+1}\right)^{n-3}}{\frac{q-1}{q+1}}\right)\right)\\
&= (q+1)^{n-3}\left((-q-3)-\frac{4}{q-1}\left(1-\left(\frac{2}{q+1}\right)^{n-3}\right)\right)\\
&= (-q-3)(q+1)^{n-3} - \frac{4\left((q+1)^{n-3}-2^{n-3}\right)}{q-1}\\
&= (-q-3)(q+1)^{n-3} - \frac{4(q+1)^{n-3}-2^{n-1}}{q-1}\\
&= \frac{-(q+3)(q-1)(q+1)^{n-3}-4(q+1)^{n-3}+2^{n-1}}{q-1}\\
&= \frac{-(q+1)^{n-3}((q+3)(q-1)-4)+2^{n-1}}{q-1}\\
&= \frac{-(q+1)^{n-3}(q+1)^2+2^{n-1}}{q-1} = \frac{-(q+1)^{n-1}+2^{n-1}}{q-1}.
\end{aligned}
$$

□

5.4.4 Results

Lemma 5.4.14. *Let $\Omega = \mathrm{SU}_{2^{n-1}}(q)$ with $q = p^e$, let $G = (2, q-1)^{\cdot}\Omega^{-}_{2n}(q)$ be an $\mathcal{S}_2^*$-candidate subgroup of Ω, with G the action group of the half-spin representation, and let $S = N_{\Omega}(G)$. Then we have:*

(i) If q is even then $S = G$. We have a single Ω-class of subgroups isomorphic to S, with trivial class stabiliser in $\mathrm{CGU}_{2^{n-1}}(q)$.

(ii) If $q \equiv 1 \mod 4$ and $n > 3$ then $S = G.2$ with the 2 automorphism induced by δ_G. We have two Ω-classes of subgroups isomorphic to S, with trivial class stabiliser in $\mathrm{CGU}_{2^{n-1}}(q)$.

(iii) If $q \equiv 3 \mod 4$ and $q \not\equiv -1 \mod 2^{n-2}$, then $S = G.4$ with the 4 automorphism induced by δ_G. We have $(2^{n-1}, q+1)$ Ω-classes of subgroups isomorphic to S, with trivial class stabiliser in $\mathrm{CGU}_{2^{n-1}}(q)$.

(iv) If $q \equiv 2^{n-2} - 1 \mod 2^{n-1}$ then $S = G.2$, with the 2 automorphism induced by $\delta_G^2 = \delta_G'$. We have 2^{n-3} Ω-classes of subgroups isomorphic to S, with class stabiliser in $\mathrm{CGU}_{2^{n-1}}(q)$ given by $\langle \delta_\Omega^{2^{n-3}} \rangle$, with $\delta_\Omega^{2^{n-3}}$ inducing δ_G, an automorphism of order 4.

(v) If $q \equiv -1 \mod 2^{n-1}$ and $n > 3$ then $S = G$. We have 2^{n-3} Ω-classes of subgroups isomorphic to S, with class stabiliser in $\mathrm{CGU}_{2^{n-1}}(q)$ given by $\langle \delta_\Omega^{2^{n-3}} \rangle$, with $\delta_\Omega^{2^{n-3}}$ inducing δ_G, an automorphism of order 4.

In all cases, ϕ_Ω induces ϕ_G.

Proof. The fact that G preserves a unitary form follows from Lemma 5.4.7, and the shape of the unitary form f is from Lemma 5.4.9. It follows from Corollary 5.4.12 that the last entry of the diagonal matrix $d_\delta^{(n)}$ is

$$\nu^{\frac{-q^{n-1}+1}{q-1}} = \nu^{-(q+1)(q^{n-3}+q^{n-5}+\ldots+q^2+1)} \in \mathbb{F}_q$$

hence both the first and last diagonal entries of $d_\delta^{(n)}$ are fixed by the field automorphism σ, so that $d_\delta^{(n)}$ rescales the form by $\nu^{-(q+1)(q^{n-3}+q^{n-5}+\ldots+q^2+1)}$. Similarly the rescaled matrix $\nu^m d_\delta^{(n)}$ scales the form by $\nu^{-(q+1)(-m+q^{n-3}+q^{n-5}+\ldots+q^2+1)}$. This is equal to 1 if and only if m is chosen such that $m \equiv q^{n-3} + q^{n-5} + \ldots + q^2 + 1 \equiv \frac{n-1}{2} \mod q-1$; hence this happens if and only if it is rescaled by an element of the form $\nu^{\frac{n-1}{2}+k(q-1)}$ for some integer k (which we may assume without loss of generality is such that $0 \leq k \leq q$).

Thus, we define $d_\delta^{(n)}(k) := d_\delta^{(n)} \nu^{\frac{n-1}{2}+k(q-1)}$ and set $a_n(k) = \det d_\delta^{(n)}(k) = a_n + (\frac{n-1}{2} + k(q-1))2^{n-1}$, with a_n such that $\det d_\delta^{(n)} = \nu^{a_n}$ as in Lemma 5.4.13. Then for every integer k, the rescaled matrix $d_\delta^{(n)}(k)$ preserves the form f_n (and the elements $d_\delta^{(n)}(k)$ are the only scalar multiples of $d_\delta^{(n)}$ which preserve the form), and has determinant 1 if and only if $a_n(k) \equiv 0 \mod q^2 - 1$.

We first consider a_n modulo q^2-1. Below, we will use the fact that $(q-1)^i \equiv (-2)^{i-1}(q-1) \mod q^2-1$ for $i \geq 1$, which is a straightforward proof by induction. We also make use of the binomial theorem; recall also that a well-known consequence of this is that $0 = \sum_{j=0}^{m} \binom{m}{i}(-1)^i$.

$$
\begin{aligned}
a_n &= -\frac{(q+1)^{n-1} - 2^{n-1}}{q-1} \\
&= -\frac{-2^{n-1} + \sum_{i=0}^{n-1} \binom{n-1}{i}(q-1)^i 2^{n-1-i}}{q-1} \\
&= -\frac{\sum_{i=1}^{n-1} \binom{n-1}{i}(q-1)^i 2^{n-1-i}}{q-1} \\
&= -\sum_{i=1}^{n-1} \binom{n-1}{i} (q-1)^{i-1} 2^{n-1-i} \\
&= -(n-1)2^{n-2} - \sum_{i=2}^{n-1} \binom{n-1}{i} (q-1)^{i-1} 2^{n-1-i} \\
&\equiv -(n-1)2^{n-2} - \sum_{i=2}^{n-1} \binom{n-1}{i} (-2)^{i-2}(q-1) 2^{n-1-i} \\
&= -(n-1)2^{n-2} - (q-1)2^{n-3} \sum_{i=2}^{n-1} \binom{n-1}{i} (-1)^i \\
&= -(n-1)2^{n-2} - (q-1)(n-2)2^{n-3}.
\end{aligned}
$$

Hence, again working modulo $q^2 - 1$ we have:

$$
\begin{aligned}
a_n(k) &= a_n + \left(\frac{n-1}{2} + k(q-1)\right) 2^{n-1} \\
&\equiv -(n-1)2^{n-2} - (q-1)(n-2)2^{n-3} + (n-1)2^{n-2} + k(q-1)2^{n-1} \\
&= -(q-1)(n-2)2^{n-3} + k(q-1)2^{n-1} \\
&= 2^{n-3}(q-1)(2+4k-n).
\end{aligned}
$$

Hence $d_\delta^{(n)}(k)$ has determinant 1 if and only if $2^{n-3}(2+4k-n) \equiv 0 \mod q+1$. Write $q+1 = 2^s r$ with r odd. Since 4 and r are coprime, 4 is a generator of the additive group $\mathbb{Z}/r\mathbb{Z}$, so that there exists a k such that $2 - n + 4k \equiv 0 \mod r$. If $s \leq n-3$ (i.e. $q \not\equiv -1 \mod 2^{n-2}$) then $2^s | 2^{n-3}$ and $r | (2+4k-n)$ so that with the prior choice of k, $a_n(k) \equiv 0 \mod q^2-1$.

Otherwise we have $q \equiv -1 \mod 2^{n-2}$ (in particular this is the only possibility when $n = 3$). Since $\nu_2(q+1) \geq n-2$ and $\nu_2(2^{n-3}(2+4k-n)) = n-3$, it follows that there is no choice of k for which $a_n(k) \equiv 0 \mod q^2-1$.

We can perform a similar computation for $\delta'_G = \delta_G^2$. For this, we require

$d_\delta^{(n)2}$ to be realisable over $\mathrm{SU}_{2^{n-1}}(q)$. In particular, we only require $d_\delta^{(n)}$ to scale the form f to $\pm f$, meaning we can rescale by an element of the form $\nu^{\frac{n-1}{2}+k\frac{q-1}{2}}$. Otherwise the computation is similar to before, and we obtain that δ'_G is realisable over $\mathrm{SU}_{2^{n-1}}(q)$ if and only if $q \not\equiv -1 \mod 2^{n-1}$. Hence we can realise δ'_G but not δ_G over $\mathrm{SU}_{2^{n-1}}(q)$ if and only if $q \equiv 2^{n-2} - 1 \mod 2^{n-1}$. Using the fact that δ'_G is outer precisely when $q \equiv 3 \mod 4$, and that δ_G^4 is always inner, we obtain the class stabilisers and the (conjugacy class of) the diagonal automorphisms of Ω which induce them.

We next consider the field automorphism ϕ_Ω. By Proposition 4.1.25 it follows that if V denotes the spin module corresponding to G, then $V^{\phi_\Omega} \cong {}^{\phi_G}V$. In the notation of Lemma 3.2.16 we can thus take $\alpha = \phi_G$ and $\beta = \phi_\Omega$. The matrix B is the antidiagonal form from Lemma 5.4.9, $L = I_{2^{n-1}}$ and $\lambda = \kappa = 1$, and so it follows from Lemma 3.2.16 that ϕ_Ω induces ϕ_G.

Finally, we consider the graph automorphism γ_Ω. From Proposition 4.1.25 it follows that ${}^{\gamma_G}V = {}^{\sigma_G}V = V^{\sigma_\Omega} = V^{\gamma_\Omega}$. Hence again we can apply Lemma 3.2.16 with $\alpha = \gamma_G$, $\beta = \gamma_\Omega$, and B, L, λ and κ as before; so that γ_Ω induces ϕ_Ω.

When q is even, there are no nontrivial outer diagonal automorphisms and it follows immediately that ϕ_Ω induces ϕ_G and γ_Ω induces γ_G. □

Lemma 5.4.15. *Let $\Omega = \mathrm{Sp}_{2^{n-1}}(q^2)$ with $n \equiv 2 \mod 4$ and $q = p^e$ with $p \neq 2$, let $G = 2^{\cdot}\mathrm{O}_{2n}^-(q)$ be an $\mathcal{S}_2^*$-candidate subgroup of Ω, with G the action group of the half-spin representation, and let $S = N_\Omega(G)$. Then we have $S = G$. We have a single Ω-class of subgroups isomorphic to S, with class stabiliser in $\mathrm{CSp}_{2^{n-1}}(q^2)$ gven by $\langle \delta_\Omega \rangle$ with δ_Ω inducing δ_G.*

Proof. From Lemma 5.4.7 and Remark 5.4.8 we see that G preserves the symplectic form as described in Lemma 5.2.10.

It follows from Proposition 4.1.25 and Lemma 1.7.13 that the automorphism δ induced by $d_\delta^{(n)}$ lies in $\mathrm{CSp}_{2^{n-1}}(q)$, and hence preserves the form up to a scalar. The form is antidiagonal with all entries on the antidiagonal either 1 or -1; hence we can determine the scalar by multiplying together the first and last entries (for example) in the diagonal of $d_\delta^{(n)}$. The first entry is 1, and by Corollary 5.4.12 the last entry is $\nu^{-q^{n-2}-q^{n-3}-\ldots-q-1} = \nu^{-\frac{q^{n-1}-1}{q-1}}$; hence whether $d_\delta^{(n)}$ can be rescaled to preserve the form depends on whether $\nu^{-\frac{q^{n-1}-1}{q-1}}$ is a square in $\mathbb{F}_{q^2}$ or not.

If n were odd, then $\nu^{-\frac{q^{n-1}-1}{q-1}} = \nu^{-(q+1)(q^{n-3}+q^{n-5}+\ldots+1)}$, which lies in $\mathbb{F}_q$ and hence is a square in $\mathbb{F}_{q^2}$. Since n is even, we can write

$$\nu^{-\frac{q^{n-1}-1}{q-1}} = \nu^{-q(q^{n-3}+q^{n-4}+\ldots+1)}\nu^{-1}.$$

From the case where n is odd, the first term is a square, and ν^{-1} is clearly primitive and hence not a square. Hence $\nu^{-\frac{q^{n-1}-1}{q-1}}$ cannot be a square when n is even. Thus we cannot rescale $d_\delta^{(n)}$ to preserve the form, and hence δ_Ω is induced by δ_G. $\square$

Lemma 5.4.16. *Let $\Omega = \Omega^+_{2^{n-1}}(q)$ with $n \equiv 0 \mod 4$ and $q = p^e$ with $p \neq 2$, let $G = 2^{\cdot}\mathrm{O}^-_{2n}(q)$ be an $\mathcal{S}_2^*$-candidate subgroup of Ω, with G the action group of the half-spin representation, and let $S = N_\Omega(G)$. Then we have $S = G$. We have four Ω-classes of subgroups isomorphic to S, with class stabiliser in $\mathrm{CGO}^+_{2^{n-1}}(q)$ given by $\langle\delta_\Omega\rangle$, with δ_Ω inducing δ_G.*

Proof. Lemma 5.4.7 and Remark 5.4.8 show that G preserves the orthogonal form as described in Lemma 5.2.10. As in the proof of Lemma 5.4.15, $d_\delta^{(n)}$ scales the form by a non-square, and hence δ_G is induced by a conjugate of either δ_Ω or $\delta_\Omega\gamma_\Omega$. Since δ_G has order 2 and $\delta_\Omega\gamma_\Omega$ has order 4, $\delta_\Omega\gamma_\Omega$ cannot induce δ_G, and hence δ_G is induced by δ_Ω. $\square$

Lemma 5.4.17. *Let $\Omega = \Omega^+_{2^{n-1}}(q)$ with $n > 4$ even and $q = 2^e$, let $G = \Omega^-_{2n}(q)$ be an $\mathcal{S}_2^*$-candidate subgroup of Ω, with G the action grop of the half-spin representation, and let $S = N_\Omega(G)$. Then we have $S = G$. There is a single Ω-class of subgroups isomorphic to S, with trivial class stabiliser in $\mathrm{CGO}^+_{2^{n-1}}(q)$.*

Proof. By Lemma 5.4.7, we have $G < \Omega$, with the outer automorphism group of Ω generated by ϕ_Ω. $\square$

Remark 5.4.18. When n is even we currently have no results regarding the action of the field automorphism ϕ_Ω on G.

5.5 2−generating sets

In [47], pairs of generators for the groups $\Omega^\epsilon_n(q)$ are provided in terms of explicit matrix generators, following the results in [49]. These matrix generators are given in terms of roots in the same manner as we have constructed the generators for the spin and half-spin representations; thus the same generating sets will work for the spin representations. Below we provide details of these generating sets.

We first introduce some notation. Let $x_\alpha^{(n)}(t)$ denote the generator of the spin or half-spin representation corresponding to the root α. This was denoted by $y_\alpha^{(n)}(t)$ for the half-spin representation $\mathrm{HSpin}_{2n}^+(q)$, and $z_\alpha^{(n)}(t)$ for $\mathrm{Spin}_{2n+1}^\circ(q)$ and $\mathrm{HSpin}_{2n}^-(q)$. We again use the convention $x_{\pm i}^{(n)}(t)$ for $x_{\pm\alpha_i}^{(n)}(t)$. We then make the following definitions, following [49, Theorem 3.4] and [9, Lemma 6.4.4], where $\epsilon \in \{+, \circ, -\}$ is the sign of the corresponding orthogonal group:

- ${}^\epsilon w_\alpha^{(n)}(t) := x_\alpha^{(n)}(t) x_{-\alpha}^{(n)}(-t^{-1}) x_\alpha^{(n)}(t)$.
- ${}^\epsilon w_\alpha^{(n)} := {}^\epsilon w_\alpha^{(n)}(1)$.
- ${}^\epsilon w^{(n)} := {}^\epsilon w_{\alpha_n}^{(n)}\, {}^\epsilon w_{\alpha_{n-1}}^{(n)} \ldots {}^\epsilon w_{\alpha_1}^{(n)}$.
- ${}^\epsilon h_\alpha^{(n)}(t) := {}^\epsilon w_\alpha^{(n)}(t)\, {}^\epsilon w_\alpha^{(n)}(-1)$.

Note that we use w_α and w in place of the more standard notation of n_α and n to avoid confusion with the dimension n of the classical group.

We will suppress the ϵ if it is clear from the context to which orthogonal group we are referring, and as usual we will use $w_{\pm i}^{(n)}(t)$, $w_{\pm i}^{(n)}$ and $h_{\pm i}^{(n)}(t)$ in place of $w_{\pm\alpha_i}^{(n)}(t)$, $w_{\pm\alpha_i}^{(n)}$ and $w_{\pm\alpha_i}^{(n)}(t)$ respectively. Recall also that we number our roots the opposite way from [47].

Throughout, let ν be a primitive element of $\mathbb{F}_q^*$.

5.5.1 $\mathrm{HSpin}_{2n}^+(q)$

Definition 5.5.1. The *Hamming weight* of an integer k, denoted in this section by $H(k)$, is the number of non-zero bits in the binary expansion of k.

In this section we also define $\epsilon(j) = \begin{cases} 0 \text{ if } H(j) \text{ is even,} \\ 1 \text{ if } H(j) \text{ is odd.} \end{cases}$

Lemma 5.5.2. ${}^+w^{(n)} = \sum_{j=1}^{2^{n-2}} (-1)^{H(j-1)+1} E_{2j,j} + \sum_{j=1}^{2^{n-2}} (-1)^n E_{2j-1,2^{n-2}+j}$.

Proof. Straightforward proofs and direct computations confirm the following facts:

$$\begin{aligned}
{}^+w_i^{(n)}(b) &= {}^+w_{i-1}^{(n-1)}(b) \oplus {}^+w_{i-1}^{(n-1)}(b) \text{ for } i \le n-2, \\
{}^+w_{n-1}^{(n)}(b) &= {}^+w_{n-2}^{(n-1)}(b) \oplus {}^+w_{n-1}^{(n-1)}(-b), \\
{}^+w_n^{(n)}(b) &= {}^+w_{n-1}^{(n-1)}(b) \oplus {}^+w_{n-2}^{(n-1)}(-b), \\
{}^+w_n^{(n)}(-1)\, {}^+w_{n-1}^{(n)}(-1) &= -\, {}^+w_n^{(n)}(1)\, {}^+w_{n-1}^{(n)}(1), \\
{}^+w_1^{(n)}(b) &= \sum_{j=1}^{2^{n-3}} E_{j,j} + \sum_{j=1}^{2^{n-3}} E_{2^{n-1}+1-j,2^{n-1}+1-j} \\
&\quad + b \sum_{j=1}^{2^{n-3}} E_{2^{n-2}+j,2^{n+3}+j} - b^{-1} \sum_{j=1}^{2^{n-3}} E_{2^{n-3}+j,2^{n-2}+j}.
\end{aligned}$$

It then follows that ${}^{+}w^{(n)} = \left({}^{+}w^{(n-1)} \oplus -{}^{+}w^{(n-1)}\right){}^{+}w_1^{(n)}$, and the result follows by a direct computation when $n = 3$, and by induction for larger n. $\square$

Recall that we have explicit constructions of the elements $y_i^{(n)}(b)$ in Section 5.2.2 and of the elements ${}^{+}h_i^{(n)}(b)$ in the proof of Theorem 5.2.5. In particular we obtain the explicit formulae:

- $y_{n-2}^{(n)}(b) = \bigoplus_{j=0}^{2^{n-3}-1} y_1^{(3)}(b) = \bigoplus_{j=0}^{2^{n-3}-1} (I_4 + bE_{3,2})$.
- $y_n^{(n)}(b) = \bigoplus_{j=0}^{2^{n-3}-1} y_{3-\epsilon(j)}^{(3)}\left((-1)^{\epsilon(j)}b\right)$.
- ${}^{+}h_{n-1}^{(n)}(b) = \bigoplus_{j=0}^{2^{n-3}-1} h_{3-\epsilon(j+1)}^{(3)}\left((-1)^{\epsilon(j)}b\right)$.
- ${}^{+}h_n^{(n)}(b) = \bigoplus_{j=0}^{2^{n-3}-1} h_{3-\epsilon(j)}^{(3)}\left((-1)^{\epsilon(j)}b\right)$.

Theorem 5.5.3. *[49, Theorems 3.11 and 3.13] and [47, Sections 4.1 and 4.2] Let $G = \mathrm{HSpin}_{2n}^{+}(q)$, and let $y_i^{(n)}(t)$ be as in Section 5.2.2 with $n \geq 4$. Then G is generated by:*

- ${}^{+}h_{n-1}^{(n)}(\nu)$ *and* $y_{-n}^{(n)}(1)y_{n-2}^{(n)}(1)\,{}^{+}w^{(n)}$ *if n is even and $q \neq 2$.*
- $y_n^{(n)}(1)y_{n-2}^{(n)}(1)$ *and* ${}^{+}w^{(n)}$ *if n is even and $q = 2$.*
- ${}^{+}h_n^{(n)}(\nu)$ *and* $y_n^{(n)}(1)\,{}^{+}w^{(n)}$ *if n is odd and $q \neq 2, 3$.*
- $y_n^{(n)}(1)$ *and* ${}^{+}w^{(n)}$ *if n is odd and $q = 2, 3$.*

5.5.2 $\mathrm{Spin}_{2n+1}^{\circ}(q)$

Note first that $z_i^{(n)}(b) = y_i^{(n+1)}(b)$ except for $i = n$, where instead we have $z_n^{(n)}(b) = y_n^{(n+1)}(b)y_{n+1}^{(n+1)}(b)$. Further, $y_{\pm n}^{(n+1)}(a)$ commutes with $y_{\pm(n+1)}^{(n+1)}(b)$. Hence it follows that ${}^{\circ}w_i^{(n)}(t) = {}^{+}w_i^{(n+1)}(t)$ for $i \leq n-1$, and ${}^{\circ}w_n^{(n)}(t) = {}^{+}w_n^{(n)}(t)\,{}^{+}w_{n+1}^{(n)}(t)$, so that ${}^{\circ}w^{(n)} = {}^{+}w^{(n+1)}$. Hence we can use Lemma 5.5.2 to determine the structure of ${}^{\circ}w^{(n)}$.

Some of the generators in [47] involve roots which are combinations of fundamental roots. We describe techniques for constructing the generators required.

Let $\epsilon_1, \ldots, \epsilon_n$ be a basis for an n-dimensional vector space over $\mathbb{Z}$. Then, from [47, Section 2.2] we can take our fundamental roots of $\mathfrak{B}_n$ to be:

$$\begin{aligned}
\alpha_1 &= \epsilon_1 - \epsilon_2, \\
\alpha_2 &= \epsilon_2 - \epsilon_3, \\
&\vdots \\
\alpha_{n-1} &= \epsilon_{n-1} - \epsilon_n, \\
\alpha_n &= \epsilon_n.
\end{aligned}$$

We then have an alternate characteristation of the diagonal element ${}^\circ h_\alpha(\nu)$, following [47, Section 4]; it is an element such that for any root β and scalar b, we have

$$^\circ h_\alpha(\nu)^{-1} z_\beta(b) \, ^\circ h_\alpha(\nu) = z_\beta(\nu^{\frac{-2(\alpha,\beta)}{(\alpha,\alpha)}} b)$$

where $(\ ,\ )$ denotes the standard dot product on $\mathbb{Z}^n$ with respect to the basis $\epsilon_1, \ldots, \epsilon_n$. In particular, in the case where α and β are fundamental roots, the coefficients $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ are the entries of the Cartan matrix.

We need to construct the generator ${}^\circ h_{\epsilon_1+\epsilon_n}(\nu)$ for the two-generating set. By considering inner products, we know how this element must scale $z_i(t)$ by conjugation; namely it scales $z_i(b)$ to $z_i(\nu^{v_i} b)$, where:

$$\begin{aligned}
v_1 &= -(\epsilon_1 + \epsilon_n, \alpha_1) = -(\epsilon_1 + \epsilon_n, \epsilon_1 - \epsilon_2) = -1, \\
v_2 &= -(\epsilon_1 + \epsilon_n, \alpha_2) = -(\epsilon_1 + \epsilon_n, \epsilon_2 - \epsilon_3) = 0, \\
&\vdots \\
v_{n-2} &= -(\epsilon_1 + \epsilon_n, \alpha_{n-2}) = -(\epsilon_1 + \epsilon_n, \epsilon_{n-2} - \epsilon_{n-1}) = 0, \\
v_{n-1} &= -(\epsilon_1 + \epsilon_n, \alpha_{n-1}) = -(\epsilon_1 + \epsilon_n, \epsilon_{n-1} - \epsilon_n) = 1, \\
v_n &= -(\epsilon_1 + \epsilon_n, \alpha_n) = -(\epsilon_1 + \epsilon_n, \epsilon_n) = -1.
\end{aligned}$$

It is easy to check that v_i is the sum of the entries in the i-th row of the Cartan matrix, and hence it follows that the element ${}^\circ h_1^{(n)}(\nu) \, {}^\circ h_2^{(n)}(\nu) \ldots {}^\circ h_{n-1}^{(n)}(\nu) \, {}^\circ h_n^{(n)}(\nu)$ scales the generators $z_i(b)$ in the required way, and hence

$$^\circ h_{\epsilon_1+\epsilon_n}^{(n)}(\nu) = {}^\circ h_1^{(n)}(\nu) \, {}^\circ h_2^{(n)}(\nu) \ldots {}^\circ h_{n-1}^{(n)}(\nu) \, {}^\circ h_n^{(n)}(\nu).$$

We can compute this element directly via the construction of the elements ${}^\circ h_i(\nu)$ in the proof of Theorem 5.3.5. A straightforward induction argument concludes that

$$ {}^\circ h^{(n)}_{\epsilon_1+\epsilon_n}(\nu) = \mathrm{diag}(\nu^{-1}, 1, \nu^{-1}, 1, \ldots, \nu^{-1}, 1, 1, \nu, 1, \nu, \ldots, 1, \nu). $$

We also need to construct $z^{(n)}_{\epsilon_1-\epsilon_n}(1) = z^{(n)}_{\alpha_1+\alpha_2+\ldots+\alpha_{n-1}}(1)$. We can construct this using successive commutators, since $[z^{(n)}_1(a), z^{(n)}_2(1)] = z^{(n)}_{1+2}(a)$, and more generally $[z^{(n)}_{1+2+\ldots+(i-1)}(a), z^{(n)}_i(1)] = z^{(n)}_{1+2+\ldots+i}(a)$. For the latter, note that $\alpha_1 + \alpha_2 + \ldots + \alpha_{i-1} = \epsilon_1 - \epsilon_i$ and $\alpha_i = \epsilon_i - \epsilon_{i+1}$; hence the angle between $\alpha_1 + \alpha_2 + \ldots + \alpha_{i-1}$ and α_i is the same as the angle between α_1 and α_2, and in particular this means we can use the structure constants as in Example 5.1.3.

Lemma 5.5.4. *For q even, $z^{(n)}_{\epsilon_1-\epsilon_n}(b) = I_{2^n} + \sum_{i=1}^{2^{n-2}} bE_{2^{n-1}-1+2i,2i}$.*

Proof. We sketch the proof as this is similar to previous calculations. We proceed by induction. The base case is as usual a direct computation. For larger n, we can check using standard inductive arguments that $z^{(n)}_{\epsilon_2-\epsilon_n}(b) = z^{(n-1)}_{\epsilon_1-\epsilon_{n-1}}(b) \oplus z^{(n-1)}_{\epsilon_1-\epsilon_{n-1}}(b)$; since q is even, $b = -b$. Recall that $z^{(n)}_{\epsilon_2-\epsilon_n}(b) = z^{(n)}_{2+3+\ldots+n}(b)$ and $z^{(n-1)}_{\epsilon_1-\epsilon_{n-1}}(b) = z^{(n-1)}_{1+2+\ldots+(n-1)}(b)$. We then have that

$$ z^{(n)}_{\epsilon_1-\epsilon_n}(b) = \left[z^{(n)}_1(b), z^{(n)}_{2+3+\ldots+n}(1) \right] $$

and the result follows by Corollary 5.2.3. □

Remark 5.5.5. A similar result to Lemma 5.5.4 holds for q odd, but this requires recording minus signs using Hamming weights similarly to Lemma 5.5.2. As we only need the result when q is even, we do not perform the more general calculation here.

Recall that Section 5.3.1 gives explicit formulae for the elements $z^{(n)}_i(b)$, and in particular we get:

- $z^{(n)}_n(b) = \bigoplus_{j=0}^{2^{n-2}-1} z^{(2)}_2\left((-1)^{H(i)+1}(b)\right) = \bigoplus_{j=0}^{2^{n-2}-1} I_4 + (-1)^{H(i)+1}(-E_{2,1} + E_{4,3})$.
- $z^{(n)}_{-n}(b) = z^{(n)}_n(b)^T$.

Theorem 5.5.6. *[49, Theorems 3.11 and 3.14] and [47, Section 4.5] Let $G = \mathrm{Spin}^\circ_{2n+1}(q)$, and let $z^{(n)}_i(t)$ be as in Section 5.3.1 with $n \geq 2$. Then G is generated by:*

- ${}^\circ h^{(n)}_{\epsilon_1+\epsilon_n}(\nu)$ *and* $z^{(n)}_n(1)\, {}^\circ w^{(n)}$ *if q is odd and $q \neq 3$.*

- $z_n^{(n)}(1)$ *and* ${}^{\circ}w^{(n)}$ *if* $q = 3$.
- ${}^{\circ}h^{(n)}_{\epsilon_1+\epsilon_n}(\nu)$ *and* $z^{(n)}_{\epsilon_1-\epsilon_n}(1)z^{(n)}_{-n}(1)\,{}^{\circ}w^{(n)}$ *if* q *is even and* $q \neq 2$.
- $z^{(n)}_{\epsilon_1-\epsilon_n}(1)z^{(n)}_{-n}(1)$ *and* ${}^{\circ}w^{(n)}$ *if* $q = 2$.

5.5.3 $\mathrm{HSpin}^-_{2n}(q)$

Similarly to the previous section, the structure of ${}^{-}w^{(n)}$ is the same as the structure of ${}^{+}w^{(n)}$. It follows from the proof of Theorem 5.4.5 that:

- ${}^{-}h^{(n)}_{n-1}(\nu) = \bigoplus_{j=0}^{2^{n-2}} \mathrm{diag}(\nu^{-q^{\epsilon(j)}}, \nu^{q^{\epsilon(j)}})$.
- $z^{(n)}_{n-1}(\nu) = \bigoplus_{j=0}^{2^{n-3}} z_2^{(3)}(\nu)^{(-\sigma)^{\epsilon(j)}}$, where $z_2^{(3)}(\nu) = I_4 - \nu E_{2,1} + \nu^q E_{4,3}$ and $z_2^{(3)}(\nu)^{\sigma} = I_4 + \nu^q E_{2,1} - \nu E_{4,3}$.

Theorem 5.5.7. *[49, Theorem 4.1] and [47, Section 5] Let* $G = \mathrm{HSpin}^-_{2n}(q)$, *and let* $z_i^{(n)}(t)$ *be as in Section 5.4.1 with* $n \geq 3$. *Then* G *is generated by* $z^{(n)}_{n-1}(1)\,{}^{-}w^{(n)}$ *and* ${}^{-}h^{(n)}_{n-1}(\nu)$.

5.6 Computations

In this section we provide details of some of the code produced relating to this chapter. A reminder that this code can be found at

`https://github.com/danielrogerswarwick/thesis`.

The file `spininduct` contains the following functions:

(1) The functions `spininductplus`, `spininductcirc` and `spininductminus` take as arguments `n` and `q`, and produce the generators as described in the sections above for $\mathrm{HSpin}^+_{2n}(q)$, $\mathrm{Spin}^{\circ}_{2n+1}(q)$ and $\mathrm{HSpin}^-_{2n}(q)$ respectively.

- By default, a generating set for the relevant spin or half-spin representation is produced; setting the optional argument `func` to `true` will instead return functions corresponding to generators of the fundamental roots, as maps $t \mapsto x_{\pm i}(t)$.
- Setting $q = 0$ will produce generators over the polynomial ring $\mathbb{Q}[t, u]$. This will be useful when checking the third set of the relations in the CST presentations in Theorem 5.1.1 and Theorem 5.1.7.

- Setting the optional argument `aut` to `true` will return a third sequence of generators corresponding to outer diagonal automorphism of the groups in question, rescaled if possible to realise them over the respective classical group.

(2) The functions `twoplus`, `twocirc` and `twominus` produce 2-generating sets for $\mathrm{HSpin}^{+}_{2n}(q)$, $\mathrm{Spin}^{\circ}_{2n+1}(q)$ and $\mathrm{HSpin}^{-}_{2n}(q)$ respectively as in Section 5.5.

(3) The functions `partialCST` and `twistCST` are adapted from code written by Don Taylor [54], and provide an implementation of the untwisted and twisted CST presentations respectively. Given inputs `tp` and `n` corresponding to the type and degree of the Dynkin diagram, and `X` and `Xm` being sequences of generators corresponding to positive and negative fundamental roots over a polynomial ring (as above), these functions return `true` if the third set of relations in the CST presentations are satisfied by `X` and `Xm`, and return `false` if a relation is not satisfied, including details of the first relation which is not satisfied.

(4) The functions `testplus`, `testcirc` and `testminus`, which take the argument `n`, and tests the generators produced by `spininductplus`, `spininductcirc` and `spininductminus` respectively to see if they satisfy the third set of relations in the CST presentation, by using the functions `partialCST` and `twistCST`. In particular, these functions can be used to test the base cases in Theorem 5.2.5, Theorem 5.3.5 and Theorem 5.4.5 respectively. Due to the large numbers of relations to check, these functions will run slowly for $n > 7$.

The current Magma implementation constructs the spin representations directly from the highest weight representation, and currently does not construct matrices inducing outer automorphisms of these representations. Our implementation of `spininduct` (hereafter referred to as the 'inductive' implementation) does (optionally) construct these outer automorphisms, and is typically faster than the current implementation, as the tables below indicate. The inductive implementation does, however, not currently construct a surjection from the spin representation to the standard representation, which the current implementation does. Note also that the current implementation produces the spin representation in all cases, whereas the inductive implementation produces the half-spin representations in even dimension (although it is straightforward to obtain generators of the spin representation from the generators of the half-spin representations).

Remark 5.6.1. Here we give some details regarding the tables below, which compare the speeds of the two implementations.

- We run the algorithms for a small range of choices of q and n to show the performance of both algorithms in various situations.
- To time the inductive implementation, we construct the half-spin generators using `spinduct`, combine these together to get generators for the spin group (when n is even), and then construct the corresponding group with these generators. The last part is often the most computationally intensive, especially for large n, and simply producing the list of generators is often much quicker than the times listed in the tables below.
- For $n > 20$, most computations failed to complete; this is largely due to the sizes of matrices involved, where memory restrictions make it hard to perform all but basic computations.
- The symbol † denotes computations which did not complete due to running out of memory.
- All times are given in seconds, and are the average running time of five computations (or fifty for additional accuracy if computations typically took less than a second).
- All computations were performed on a 2.4GHz PC (Intel i7) with 8GB of RAM.

Table 5.1: Time to compute $\mathrm{Spin}_n^+(q)$

ϵ	n	q	Inductive	Current
$+$	12	7	0.002	1.60
$+$	12	7^6	0.010	1.78
$+$	12	117,643	0.009	1.60
$+$	14	7	0.009	6.60
$+$	14	7^6	0.053	6.99
$+$	14	117,643	0.034	6.78
$+$	16	7	0.027	28.7
$+$	16	7^6	0.192	29.9
$+$	16	117,643	0.171	28.3
$+$	18	7	0.113	114
$+$	18	7^6	0.785	118
$+$	18	117,643	1.09	115
$+$	20	7	0.491	444
$+$	20	7^6	3.51	467
$+$	20	117,643	7.23	452

Table 5.2: Time to compute $\mathrm{Spin}_n^\circ(q)$

ϵ	n	q	Inductive	Current
$\circ$	13	7	0.003	2.68
$\circ$	13	7^6	0.016	2.81
$\circ$	13	117,643	0.007	2.65
$\circ$	15	7	0.008	11.3
$\circ$	15	7^6	0.055	11.7
$\circ$	15	117,643	0.038	11.5
$\circ$	17	7	0.027	47.2
$\circ$	17	7^6	0.215	49.0
$\circ$	17	117,643	0.196	48.5
$\circ$	19	7	0.124	188
$\circ$	19	7^6	0.866	192
$\circ$	19	117,643	1.21	199
$\circ$	21	7	0.582	699
$\circ$	21	7^6	4.34	719
$\circ$	21	117,643	7.95	715

Table 5.3: Time to compute $\mathrm{Spin}_n^-(q)$

ϵ	n	q	Inductive	Current
$-$	12	7	0.004	2.21
$-$	12	7^3	0.008	6.08
$-$	12	7^6	0.045	†
$-$	12	117,643	0.009	†
$-$	14	7	0.015	7.74
$-$	14	7^3	0.028	11.4
$-$	14	7^6	0.181	†
$-$	14	117,643	0.031	†
$-$	16	7	0.044	31.1
$-$	16	7^3	0.103	37.4
$-$	16	7^6	0.709	†
$-$	16	117,643	0.125	†
$-$	18	7	0.163	117
$-$	18	7^3	0.409	124
$-$	18	7^6	3.02	†
$-$	18	117,643	0.500	†
$-$	20	7	0.738	458
$-$	20	7^3	1.75	†
$-$	20	7^6	12.6	†
$-$	20	117,643	2.18	†

Chapter 6

Containments

6.1 Introductory notes

In this chapter we will investigate containments between the $\mathcal{S}_i$-maximal candidates of classical groups in dimensions 16 and 17 as described in Chapter 3 and Chapter 4, and the geometric-type subgroups. Recall from Definition 1.10.33 that there is no containment of groups in Class $\mathcal{S}$ inside groups in Classes $\mathcal{C}_1$, $\mathcal{C}_3$, $\mathcal{C}_5$ or $\mathcal{C}_8$. Section 6.2 discusses containments between Classes $\mathcal{S}_1$ and $\mathcal{S}_2^*$ (containments within the classes $\mathcal{S}_1$ and $\mathcal{S}_2^*$ have been discussed already), producing a list of $\mathcal{S}$-maximal candidates. Section 6.3 discusses containments between these $\mathcal{S}$-maximal candidates and the geometric-type subgroups listed in Chapter 7, thus completing the classification of the maximal subgroups of classical groups.

For notational convenience we make the following natural definition:

Definition 6.1.1. A group S is an $\mathcal{S}^*$*-maximal subgroup of* G if it is maximal within the classes $\mathcal{S}_1$ and $\mathcal{S}_2^*$.

The geometric-type maximal candidates have been classified already:

Theorem 6.1.2. *Let M be a maximal subgroup of a classical group Ω in dimension 16 or 17, with M lying in one of the Aschbacher classes $\mathcal{C}_1, \ldots, \mathcal{C}_8$. Then M is in one of the geometric tables in Chapter 7.*

Proof. Direct from the tables in [33, Chapter 3]. $\square$

We will make use of the tables of geometric-type maximal subgroups in Chapter 7 in Section 6.3.

6.2 Containments in Class $\mathcal{S}$

Note that there are no $\mathcal{S}^*$-maximal subgroups of $\mathrm{SL}^{\pm}_{17}(q)$, and no $\mathcal{S}_2^*$-maximal subgroups of $\Omega^-_{16}(q)$; hence there are no containments to check here in these cases.

6.2.1 $\mathrm{SL}_{16}(q)$

The candidate $\mathcal{S}_1$-maximal subgroups of $\mathrm{SL}_{16}(q)$ are (extensions of) $2^{\cdot}A_{12}$ with $q = p = 3$, $2^{\cdot}A_{11}$ with $q = p \equiv 1, 3, 4, 5, 9 \mod 11$ and $p \neq 3$, M_{12} with $q = p \equiv 1, 3, 4, 5, 9 \mod 11$ and $p \neq 3$, $2^{\cdot}\mathrm{L}_2(31)$ with $q = p \neq 2$ and p a square modulo 31, and $\mathrm{L}_3(3)$ with $q = p \equiv 1, 3, 9 \mod 13$ and $p \neq 3$. See Theorem 3.4.31 for details.

The candidate $\mathcal{S}_2^*$-maximal subgroups of $\mathrm{SL}_{16}(q)$ are (extensions of) $\mathrm{SL}_4(q)$ with $p = 3$, $\mathrm{L}_4(q^2)$ for any prime p, and $(2, q-1)^{\cdot}\Omega^+_{10}(q)$ for any prime p. See Theorem 4.6.1 for details.

Lemma 6.2.1. *Let $\Omega = \mathrm{SL}_{16}(q)$. Then all $\mathcal{S}_1$-maximal and $\mathcal{S}_2^*$-maximal subgroups of all almost simple extensions of $\overline{\Omega}$ are $\mathcal{S}^*$-maximal.*

Proof. All the $\mathcal{S}_2^*$-candidates have natural representations in dimension at most 10, and the dimension of the smallest nontrivial projective representation of the nonabelian composition factor of each of the $\mathcal{S}_1$-candidates is at least 11, with the exceptions of A_{12} and A_{11}, both of which have representations which are contained in $\Omega^+_{10}(q)$; however, there is no embedding of $2^{\cdot}A_{12}$ or $2^{\cdot}A_{11}$ inside $\Omega^+_{10}(q)$. Hence there are no containments of $\mathcal{S}_1$-candidates inside $\mathcal{S}_2^*$-candidates.

Conversely, Lagrange does not provide any possibility of containment of any $\mathcal{S}_2^*$-candidate inside any $\mathcal{S}_1$-candidate, typically by checking the q-part of the order of the $\mathcal{S}_2^*$-candidate and noting that the highest p-part of any $\mathcal{S}_1$-candidate subgroup is $\nu_2(2^{\cdot}A_{12}) = 10$. Noting that $\nu_q(\mathrm{L}_4(q)) = 6$, this rules out any possible containments except $\mathrm{L}_4(2^2)$ inside $2^{\cdot}A_{12}$, which can be ruled out by a direct check. □

6.2.2 $\mathrm{SU}_{16}(q)$

The candidate $\mathcal{S}_1$-maximal subgroups of $\mathrm{SU}_{16}(q)$ are (extensions of) A_{12} with $q = p = 2$, $2^{\cdot}A_{11}$ with $q = p$ and p not a square modulo 11, $4^{\cdot}M_{22}$ with $q = p = 7$, M_{12} with $q = p$ and p not a square modulo 11, $4_2^{\cdot}\mathrm{L}_3(4)$ with $q = p = 3$, $2^{\cdot}\mathrm{L}_2(31)$ with $q = p$ and p not a square modulo 31, and $\mathrm{L}_3(3)$ with $q = p \equiv 4, 10, 12 \mod 13$ or $q = p^2$ with $p \equiv 2, 5, 6, 7, 8, 11 \mod 13$. See Theorem 3.4.31 for details.

The candidate $\mathcal{S}_2^*$-maximal subgroups of $\mathrm{SU}_{16}(q)$ are $\mathrm{SU}_4(q).(q+1, 4)$ when $p = 3$, extensions of $\mathrm{L}_4(q^2)$ for any prime p, and extensions of $(2, q-1)^{\cdot}\Omega^-_{10}(q)$ for any prime p. See Theorem 4.6.2 for details.

Lemma 6.2.2. *Let $\Omega = \mathrm{SU}_{16}(q)$. Then the only containments of $\mathcal{S}_1$-maximal subgroups of $\overline{\Omega}$ in $\mathcal{S}_2^*$-maximal subgroups of $\overline{\Omega}$ are:*

(i) $A_{12} < \Omega_{10}^-(2) < \mathrm{SU}_{16}(2)$, *and A_{12} and $A_{12}.2$ are not maximal in any extension of Ω by automorphisms.*

(ii) $4^{\cdot}_2\mathrm{L}_3(4).2_2 < \mathrm{SU}_4(3).4 < \mathrm{SU}_{16}(3)$. *The group $4^{\cdot}_2\mathrm{L}_3(4).2^2$ is a maximal subgroup of extensions of $\mathrm{SU}_{16}(3)$ involving γ_Ω.*

(iii) $M_{12}.2 < \Omega_{10}^-(2).\langle\gamma_{\Omega_{10}^-(2)}\rangle < \Omega.\langle\gamma_\Omega\rangle$, *and M_{12} and $M_{12}.2$ are not maximal in any extension of Ω by automorphisms.*

All other $\mathcal{S}_1$-maximal subgroups, and all $\mathcal{S}_2^*$-maximal subgroups, of all other almost simple extensions of $\overline{\Omega}$ are $\mathcal{S}^*$-maximal.

Proof. From [12] and [32] we see that the only $\mathcal{S}_1$-candidates with a faithful irreducible representation of dimension ≤ 10 are:

(i) A_{12}, which has a 10-dimensional irreducible representation in characteristic 2 with Schur indicator $+$.

(ii) $4^{\cdot}_2\mathrm{L}_3(4).2_2$, which has two weakly-equivalent 4-dimensional irreducible representations in characteristic 3 with Schur indicator $\circ$.

(iii) M_{12} has two weakly-equivalent 10-dimensional irreducible representations in characteristic 2 with Schur indicator $+$; although M_{12} is not maximal in Ω, $M_{12}.2$ is $\mathcal{S}_1$-maximal in $\Omega.\langle\gamma\rangle$.

For $G = A_{12}$, we see from [8, Table 8.69] that there is an abstract containment $A_{12} < \Omega_{10}^-(q)$. Computations in `s1s2cont` show that there is a containment $A_{12} < \Omega_{10}^-(q)$. The representation of A_{12} is stabilised in Ω by γ_Ω, which induces $\gamma_{\Omega_{10}^-(q)}$ on $\Omega_{10}^-(q)$, which rules out the possibility of a type 1 novelty. Also from [8, Table 8.69], we see that $\gamma_{\Omega_{10}^-(q)}$ stabilises A_{12} and hence induces the unique nontrivial outer automorphism on A_{12}, as γ_Ω does, ruling out the possibility of a type 2 novelty. Hence no extension of Ω contains an extension of A_{12} as a maximal subgroup.

Similarly, from [8, Table 8.11] we see that there are abstract containments of $4^{\cdot}_2\mathrm{L}_3(4) < \mathrm{SU}_4(3)$, and $4^{\cdot}_2\mathrm{L}_3(4).2_2 < \mathrm{SU}_4(3).\delta^2$. Since the $\mathcal{S}_2^*$-candidate is $\mathrm{SU}_4(3).\delta$, we have an abstract containment here, and computations in `s1s2cont` show that this is a containment. The class stabiliser of $4^{\cdot}_2\mathrm{L}_3(4).2_2$ in Ω is γ_Ω, and γ_Ω also stabilises $\mathrm{SU}_4(3).\delta$; however γ_Ω induces $\gamma_{\mathrm{SU}_4(3)}$ by Lemma 4.4.6, and from [8, Proposition

4.7.5] we see that $\gamma_{\mathrm{SU}_4(3)}$ does not stabilise $4_2^{\cdot}\mathrm{L}_3(4).2_2$; hence we have a type 2 novelty here. Thus $4_2^{\cdot}\mathrm{L}_3(4).2_2$ is maximal amongst extensions of Ω involving γ_Ω, and is not maximal otherwise.

Similar computations also show there is a containment $M_{12} < \Omega_{10}^-(2)$ (indeed, this follows from the abstract containment $M_{12} < A_{12}$ and the Brauer character table of M_{12} from [32, p.74]). From [8, Table 8.69] we see that $M_{12}.2$ is maximal under $\Omega_{10}^-(2).\langle\gamma_{\Omega_{10}^-(2)}\rangle$ giving the chain of subgroups $M_{12}.2 < \Omega_{10}^-(2).\langle\gamma_{\Omega_{10}^-(2)}\rangle < \Omega.\langle\gamma_\Omega\rangle$ (it follows from Lemma 4.3.29 that γ_Ω induces $\gamma_{\Omega_{10}^-(2)}$). Since $M_{12}.2$ has no outer automorphisms there is no possibility of any novelty subgroups.

Lagrange rules out any possible containment of $\mathcal{S}_2^*$-candidates in $\mathcal{S}_1$-candidates in a similar manner to Lemma 6.2.1 by considering q-parts and checking by hand the small number of cases this does not eliminate. □

6.2.3 $\mathrm{Sp}_{16}(q)$

The candidate $\mathcal{S}_1$-maximal subgroups of $\mathrm{Sp}_{16}(q)$ are (extensions of) A_{18} for $p = 2$, $2^{\cdot}A_8$ for $p = 7$ and two representations of $\mathrm{SL}_2(17)$; $\mathrm{SL}_2(17)_1$ and $\mathrm{SL}_2(17)_2$, both with $p \neq 2, 3, 17$. See Theorem 3.4.28 for details.

The candidate $\mathcal{S}_2^*$-maximal subgroups of $\mathrm{Sp}_{16}(q)$ are (extensions of) $\mathrm{SL}_2(q)$ for $p \geq 17$ and $\mathrm{Sp}_4(q)$ for $p \neq 2, 5$. See Theorem 4.6.3 for details.

Lemma 6.2.3. *Let $\Omega = \mathrm{Sp}_{16}(q)$. Then all $\mathcal{S}_1$-maximal subgroups of all almost simple extensions of $\overline{\Omega}$ are $\mathcal{S}^*$-maximal.*

Proof. For the $\mathcal{S}_2^*$-candidate group $\mathrm{SL}_2(q)$, we require $p \geq 17$ ruling out containments of $2^{\cdot}A_8$ and $A_{18}.2$, and [12] rules out $\mathrm{SL}_2(17)$ having a 2-dimensional faithful representation in non-defining characteristic. Hence no $\mathcal{S}_1$-candidates can be contained in $\mathrm{SL}_2(q)$.

Restrictions on p also rule out containment of $A_{18}.2$ in $\mathrm{Sp}_4(q)$; [12] and [32] eliminate the possibility of containment of the remaining $\mathcal{S}_1$-candidates in $\mathrm{Sp}_4(q)$. □

Lemma 6.2.4. *Let $\Omega = \mathrm{Sp}_{16}(q)$. Then all $\mathcal{S}_2^*$-maximal subgroups of all almost simple extensions of $\overline{\Omega}$ are $\mathcal{S}^*$-maximal.*

Proof. We consider each of the $\mathcal{S}_1$-candidates in turn.

The group A_{18} occurs only in characteristic 2, whereas there are no $\mathcal{S}_2^*$-candidates in characteristic 2. Similarly $2^{\cdot}A_8$ occurs only in characteristic 7, and the only $\mathcal{S}_2^*$-candidate in characteristic 7 is $\mathrm{Sp}_4(7^e)$, which cannot be contained in $2^{\cdot}A_8$ by Lagrange.

The remaining candidates are representations of $\mathrm{SL}_2(17)$ with $p \neq 2, 3, 17$. There is no containment of $\mathrm{SL}_2(q)$ in $\mathrm{SL}_2(17)$ when $p \geq 19$ by considering minimal primitive permutation representations. Lagrange rules out any containment of $\mathrm{Sp}_4(q)$ inside $\mathrm{SL}_2(17)$. $\square$

6.2.4 $\Omega_{16}^+(q)$

The candidate $\mathcal{S}_1$-maximal subgroups of $\Omega_{16}^+(q)$ are (extensions of) A_{17} with $q = p \neq 17$ a square modulo 17, $2^{\cdot}A_{11}$ with $q = p = 11$, $2^{\cdot}A_{10}$ with $q = p \neq 2, 5$, A_{10} with $q = p = 2$, M_{12} with $q = p = 11$, $2^{\cdot}\mathrm{Sz}(8)$ with $q = p = 13$, $\mathrm{L}_3(3)$ with $q = p = 13$ and two representations of $\mathrm{L}_2(17)$; $\mathrm{L}_2(17)_1$ with $q = p \neq 17$ a square modulo 17, and $\mathrm{L}_2(17)_2$ with $p \neq 3, 17$ a square modulo 17, and $q = p$ if $p \equiv \pm 1 \mod 9$ and $q = p^3$ otherwise. See Theorem 3.4.29 for details.

The candidate $\mathcal{S}_2^*$-maximal subgroups of $\Omega_{16}^+(q)$ are $\mathrm{L}_2(q^4).4$ for any prime p, $\mathrm{Sp}_4(q^2).2$ for any prime p, $2^{\cdot}\Omega_9^\circ(q)$ for $p \neq 2$, and $\mathrm{Sp}_8(q)$ for $p = 2$. See Theorem 4.6.4 for details.

Lemma 6.2.5. *Let $\Omega = \Omega_{16}^+(q)$. Then the only containments of $\mathcal{S}_1$-maximal subgroups of $\overline{\Omega}$ in $\mathcal{S}_2^*$-maximal subgroups of $\overline{\Omega}$ are:*

(i) $\mathrm{L}_2(17)_1 < 2^{\cdot}\Omega_9^\circ(p) < \Omega_{16}^+(p)$ *for odd primes $p \neq 17$, and* $\mathrm{L}_2(17)_1$ *is an $\mathcal{S}_2^*$-maximal candidate only for extensions of Ω which contain γ_Ω.*

(ii) $\mathrm{L}_2(17)_1 < \mathrm{Sp}_8(2) < \Omega_{16}^+(2)$*, and* $\mathrm{L}_2(17)_1.2$ *is an $\mathcal{S}_2^*$-maximal subgroup of* $\Omega_{16}^+(2).\gamma$*.*

(iii) $A_{10}.2 < \mathrm{Sp}_8(2) < \Omega_{16}^+(2)$*, and $A_{10}.2$ is not maximal in any almost simple extension of Ω.*

All other $\mathcal{S}_1$-maximal subgroups of all almost simple extensions of $\overline{\Omega}$ are $\mathcal{S}^*$-maximal.

Proof. First suppose that q is odd. Note that the $\mathcal{S}_2^*$-candidates all have natural representations of dimension at most 9. From the character tables in [12] and [32] we see that the only $\mathcal{S}_1$-candidate with a faithful representation of dimension at most 9 in odd characteristic is $\mathrm{L}_2(17)$, which has a 9-dimensional ordinary character. Hence the only possible containments we can have is of a representation of $\mathrm{L}_2(17)$ inside $2^{\cdot}\Omega_9^\circ(q)$.

From the (ordinary) character table of $\mathrm{L}_2(17)$ in [12], we see that there is only one conjugacy class of irreducible representations of $\mathrm{L}_2(17)$ inside $\Omega_9^\circ(p)$ (when p is a square mod 17). Consider the spin representation $\rho : 2^{\cdot}\Omega_9^\circ(p) \to \Omega_{16}^+(p)$. We have that $\mathrm{L}_2(17)\rho < \Omega_{16}^+(p)$ and thus gives a 16-dimensional representation of

$\mathrm{L}_2(17)$. However, from the construction of the spin representation in Chapter 5, we know that we can obtain the representation over $\mathbb{F}_p$ as the p-modular reduction of matrices over $\mathbb{Q}$ (indeed over $\mathbb{Z}$); thus in particular the representation $\mathrm{L}_2(17)\rho$ must have an integer-valued character. The only 16-dimensional faithful integer-valued character of $\mathrm{L}_2(17)$ is χ_4 (in the notation of [12]), which corresponds to the representation of $\mathrm{L}_2(17)_1$. Hence we have a containment $\mathrm{L}_2(17)_1 < 2{}^{\cdot}\Omega_9^{\circ}(p)$. The class stabiliser of $G = \mathrm{L}_2(17)_1$ in Ω is γ_Ω, and the class stabiliser of $2{}^{\cdot}\Omega_9^{\circ}(p)$ in Ω is δ_Ω; hence $\mathrm{L}_2(17)_1.2$ is maximal in extensions of Ω containing γ_Ω, and otherwise is not maximal.

Since there is only one 9-dimensional character of $\mathrm{L}_2(17)$, hence only one embedding of $\mathrm{L}_2(17)$ inside $2{}^{\cdot}\Omega_9^{\circ}(p)$, there is no containment of $\mathrm{L}_2(17)_2$ inside an $\mathcal{S}_2^*$-maximal candidate.

Next, suppose that q is even. All $\mathcal{S}_2^*$-candidates have natural representations in dimension at most 8, and from [32] and [56] we see that the only $\mathcal{S}_1$-candidates with faithful representations of dimension at most 8 in even characteristic are $\mathrm{L}_2(17)$ with two weakly-equivalent 8-dimensional representations preserving a symplectic form (and character ring generated by b_{17}, which exists over $\mathbb{F}_2$), and $A_{10}.2$, which also has an 8-dimensional representation preserving a symplectic form (with character ring $\mathbb{Z}$). Thus we have abstract containments $\mathrm{L}_2(17) < \mathrm{Sp}_8(q)$ and $A_{10}.2 < \mathrm{Sp}_8(q)$ for q even.

We first consider $\mathrm{L}_2(17) < \mathrm{Sp}_8(q)$. As for the case when q is odd, since the character ring of the spin representation $\rho : \mathrm{Sp}_8(2) \to \Omega_{16}^+(2)$ is integer-valued, it follows that the restriction of ρ to $\mathrm{L}_2(17)$ gives the representation $\mathrm{L}_2(17)_1$, so that we have a containment $\mathrm{L}_2(17)_1 < \mathrm{Sp}_8(q)$. The representation $\mathrm{L}_2(17)_1$ has class stabiliser $\langle\gamma_\Omega\rangle$ and $\mathrm{Sp}_8(2)$ has trivial class stabiliser; hence again $\mathrm{L}_2(17)_1.2$ is maximal in extensions of $\mathrm{Sp}_8(2)$ containing γ_Ω and otherwise is not maximal.

We next consider $A_{10}.2 = S_{10}$. Computations in `a10d2calc` show that we have a containment of S_{10} inside the spin representation $\mathrm{Sp}_8(2)$, and since S_{10} has trivial class stabiliser inside $\Omega_{16}^+(2)$, the group A_{10} is not $\mathcal{S}_2^*$-maximal in any almost simple extension of $\Omega_{16}^+(2)$. □

Lemma 6.2.6. *The only containment of an $\mathcal{S}_2^*$-maximal subgroup of $\overline{\Omega}$ in an $\mathcal{S}_1$-maximal subgroup of $\overline{\Omega}$ when $\Omega = \Omega_{16}^+(q)$ is of $\mathrm{L}_2(16).4 < A_{17}$ when $q = 2$, and $\mathrm{L}_2(16).4$ is not maximal in any almost simple extension of Ω. All other $\mathcal{S}_2^*$-maximal subgroups of all almost simple extensions of $\overline{\Omega}$ are $\mathcal{S}^*$-maximal.*

Proof. There are no possible containments involving the $\mathcal{S}_2^*$-candidate $2{}^{\cdot}\Omega_9^{\circ}(q)$, as the q-part of this group is larger than the q-part of the $\mathcal{S}_1$-candidates for any choice of q. Similarly by considering q-parts we can rule out containments involving $\mathrm{Sp}_8(q)$ when q is even, and those involving $\mathrm{Sp}_4(q^2)$ when q is odd.

Consider the $\mathcal{S}_2^*$-candidate $G = \mathrm{Sp}_4(q^2)$ when q is even. The group G is perfect and has no subgroup of index ≤ 17, so that we cannot have a containment of G in any extension of any alternating group which is also an $\mathcal{S}_1$-candidate. The only other $\mathcal{S}_1$-candidate in even characteristic is $\mathrm{L}_2(17)$, which also cannot contain $\mathrm{Sp}_4(q^2)$ for any even q. Hence there are no containments involving $\mathrm{Sp}_4(q^2)$.

The $\mathcal{S}_2^*$-candidate $\mathrm{L}_2(q^4).4$ has a permutation representation on 17 points when $q = 2$, so we may have a containment $\mathrm{L}_2(16).4 < A_{17}$; otherwise $\mathrm{L}_2(q^4).4$ is not contained in any $\mathcal{S}_1$-candidate extension of an alternating group. Lagrange also rules out containments between $\mathrm{L}_2(q^4)$ and any of the remaining $\mathcal{S}_1$-candidates.

There is a natural embedding of $G = \mathrm{L}_2(16).4 < A_{17}$. From the file `s1s2cont` we see that this embedding gives rise to a 16-dimensional irreducible representation of $\mathrm{L}_2(16).4$. (The computer calculation also verifies the conjectured behaviour of the 4 automorphism of $\mathrm{L}_2(16)$ in this case). From [32] we see that there are two classes of irreducible 16-dimensional representations of $\mathrm{L}_2(16).4$ in characteristic 2, which differ on whether the subgroup $\mathrm{L}_2(16)$ of $\mathrm{L}_2(16).4$ acts irreducibly on the 16-dimensional module. Both the module from this natural embedding and the $\mathcal{S}_1$-maximal candidate act irreducibly on the module via the subgroup $\mathrm{L}_2(16)$, so there is a containment $\mathrm{L}_2(16).4 < A_{17}$. Since $\mathrm{L}_2(16).4$ has trivial outer automorphism group, it follows that there are no nontrivial almost simple extensions of $\mathrm{L}_2(16).4$, and hence $\mathrm{L}_2(16).4$ is not maximal in any almost simple extension of Ω. □

6.2.5 $\Omega_{17}^{\circ}(q)$

The candidate $\mathcal{S}_1$-maximal subgroups of $\Omega_{17}^{\circ}(q)$ are (extensions of) A_{19} for $p = 2$, A_{18} for $p \neq 2, 3, 19$ and three representations of $\mathrm{L}_2(16)$; namely $\mathrm{L}_2(16)_1$ for $p \neq 2, 3$, $\mathrm{L}_2(16)_2$ for $p \neq 2, 5$ and $\mathrm{L}_2(16)_3$ for $p \neq 2, 3, 5$. See Theorem 3.3.13 for details.

The only candidate $\mathcal{S}_2^*$-maximal candidate subgroup is $\mathrm{L}_2(q).2$ for $p \geq 17$. See Theorem 4.6.6 for details.

Lemma 6.2.7. *Let $\Omega = \Omega_{17}^{\circ}(q)$. The only containment of an $\mathcal{S}_2^*$-maximal subgroup of $\overline{\Omega}$ in an $\mathcal{S}_1$-maximal subgroup of $\overline{\Omega}$ is when $q = 17$, when we have* $\mathrm{L}_2(17).2 < \mathrm{A}_{18}.2$. *The $\mathcal{S}_2^*$-candidate* $\mathrm{L}_2(17).2$ *is not maximal in any almost simple extension of $\overline{\Omega}$. All other $\mathcal{S}_2^*$-maximal subgroups of all almost simple extensions of $\overline{\Omega}$ are $\mathcal{S}^*$-maximal.*

Proof. The only $\mathcal{S}_2^*$-subgroup is $\mathrm{L}_2(q).2 = \mathrm{PGL}_2(q)$, which has a primitive permutation representation on $q+1$ points and no smaller permutation representations. The $\mathcal{S}_1$-candidates are $\mathrm{A}_{19}.2$, $\mathrm{A}_{18}.2$ and extensions of $\mathrm{L}_2(16)$, which all have faithful permutation representations on 19 points, ruling out any containment of $\mathcal{S}_2^*$-candidates in $\mathcal{S}_1$-candidates when $q \geq 19$. Hence the only situation where we might have a containment in this direction is when $q = 17$. Here we have abstract containments $\mathrm{PSL}_2(17) < \mathrm{A}_{18}$ and $\mathrm{PGL}_2(17) < \mathrm{A}_{18}.2$. We can find the 17-dimensional character of $\mathrm{PGL}_2(17)$ in [32], which we can see has character equal to the standard permutation representation of $\mathrm{PGL}_2(17)$ in A_{18} minus the trivial character, which is the same as the 17-dimensional absolutely irreducible character of $\mathrm{A}_{18}.2$. Hence we have a containment in this case. Further, since $\mathrm{PGL}_2(17)$ has trivial outer automorphism group, it also has trivial class stabiliser and hence is not maximal in any almost simple extension of $\overline{\Omega}$. $\square$

Lemma 6.2.8. *Let $\Omega = \Omega_{17}^{\circ}(q)$. Then all $\mathcal{S}_1$-maximal subgroups of all almost simple extensions of $\overline{\Omega}$ are $\mathcal{S}^*$-maximal.*

Proof. The only $\mathcal{S}_2^*$-subgroup of Ω is $G := \mathrm{PGL}_2(q)$. If there is a containment of an $\mathcal{S}_1$-candidate in G, it must in particular have a 2-dimensional projective representation. The groups A_{18} and A_{19} do not have a 2-dimensional representation in any characteristic, and $\mathrm{L}_2(16)$ does not have a 2-dimensional representation in odd characteristic; hence there are no possible containments. $\square$

6.3 Containments between $\mathcal{S}^*$ and geometric type subgroups

From the previous sections we now have a complete list of $\mathcal{S}^*$-maximal candidate subgroups and geometric-type maximal subgroups. To complete the classification of all maximal subgroups of the 16- and 17-dimensional classical groups, we need to check containments between these two lists.

6.3.1 Preliminary results

Containments of $\mathcal{S}^*$-candidates in geometric-type subgroups

When considering the $\mathcal{C}_2$-subgroups, we will make use of the theory of induced characters from Section 1.7.3.

Suppose G is a classical group in dimension n and characteristic p and we have a $\mathcal{C}_2$-subgroup C of G with $C = \hat{G} \wr S_t$, the wreath product of a classical group $\hat{G}$ in dimension $\frac{n}{t}$ with S_t, decomposing the natural module of G into a direct sum of t subspaces of dimension $\frac{n}{t}$. Suppose we also have an $\mathcal{S}^*$-candidate subgroup S with corresponding n-dimensional character τ and representation ρ_τ and we are interested in investigating the containment of $S\rho_\tau$ in C. If there were such a containment, then we would require S to have a subgroup H of index t, with a $\frac{n}{t}$-dimensional character χ and representation ρ_χ such that $H\rho_\chi < \hat{G}$ and $\chi^G = \tau$. Given the construction of χ^G, it is clear that the character ring of $\tau = \chi^G$ is contained in the character ring of χ, but it is possible for τ to have a strictly smaller character ring in certain characteristics, so we additionally require that the character ring of χ in characteristic p is no larger than the character ring of τ. If this is the case then it follows straightforwardly that this condition is necessary and sufficient for a containment of $S\rho_\tau$ in C.

Containments of geometric-type subgroups in $\mathcal{S}^*$-candidates

It is a consequence of [33, Theorem 8.1.1] that there are no containments of geometric-type subgroups in $\mathcal{S}^*$-candidates that have not already been ruled out in the tables in [33]; however, in the cases we are considering these proofs are not especially difficult, and so we include them here for completeness.

We describe some commonly-used methods for ruling out some potential containments.

Recall Definition 1.5.6 of $\nu_q(n)$. We will often be interested in the q-part of the order of a group G, and will consequently write $\nu_q(G)$ in place of $\nu_q(|G|)$.

If Ω is a classical group in characteristic p, in many cases counting the p-part of the geometric-type subgroup is sufficient to rule out possible containments in any $\mathcal{S}^*$-candidate subgroup. This is usually effective when dealing with $\mathcal{C}_1$ and $\mathcal{C}_3$ candidates.

The below is a consequence of Lemma 2.3.24.

Corollary 6.3.1. *Let q be odd, and $e \geq 1$ be an integer. Then*

(i) If e is odd then $\nu_2(q^e - 1) = \nu_2(q - 1)$.

(ii) If e is even then $\nu_2(q^e - 1) = \max(\nu_2(q + 1), \nu_2(q - 1)) + \nu_2(e)$.

Proof. When e is odd, note that $\frac{q^e-1}{q-1} = 1 + q + \ldots + q^{e-1}$ is odd, so the result follows.

When e is even, let $r = \nu_2(e)$, so that $e = 2^r s$ with s odd. Also let $t = \max(\nu_2(q + 1), \nu_2(q - 1))$. Then by Lemma 2.3.24 it follows that $q^e = (q^{2^r})^s \equiv 1 \mod 2^{t+r}$, so that $\nu_2(q^e - 1) \geq t + r$. We also have that $q \equiv \pm 1 \mod 2^t$ and $q \not\equiv \pm 1 \mod 2^{t+1}$ by definition of t. Since s is odd, it follows that $q^s \equiv q \mod 2^{t+1}$, and thus it also follows from Lemma 2.3.24 that $(q^s)^{2^r} \equiv q^{2^r} \not\equiv 1 \mod 2^{t+r+1}$, so that $\nu_2(q^e - 1) = t + r$. $\square$

Corollary 6.3.2. *Suppose q and e are odd. Then $\nu_2(q^e + 1) = \nu_2(q + 1)$.*

Proof. Since $q^{2e} - 1 = (q^e + 1)(q^e - 1)$ and ν_2 is such that $\nu_2(ab) = \nu_2(a) + \nu_2(b)$, we can use Corollary 6.3.1 to establish that

$$\nu_2(q^e + 1) = 1 - \nu_2(q - 1) + \max(\nu_2(q - 1), \nu_2(q + 1))$$

and since precisely one of $\nu_2(q-1)$ and $\nu_2(q+1)$ is equal to 1, the result follows. $\square$

The above results provide another method for ruling out containments via Lagrange when q is odd, by considering 2-parts. When q is even the below corollary of Zsigmondy's theorem will also be useful (see for example [4] for a proof of Zsigmondy's theorem).

Lemma 6.3.3 (Bang's theorem)**.** *If $n > 1$, $n \neq 6$, then $2^n - 1$ has a prime factor which does not divide any number of the form $2^k - 1$ for $k < n$.*

The below results will be useful when the geometric-type group under consideration is a classical group in comparatively large dimension (for instance in Classes $\mathcal{C}_5$ or $\mathcal{C}_8$) or contains a direct product of a number of smaller classical groups (for instance in Classes $\mathcal{C}_2$ or $\mathcal{C}_7$).

Definition 6.3.4. Let G be a finite group, and p be prime. Then

$$R_p(G) := \min\{n : G \text{ is isomorphic to a subgroup of } \mathrm{PGL}_n(\overline{\mathbb{F}_p})\}.$$

Lemma 6.3.5. *[41] Let S be a nonabelian simple classical group in dimension d over $\mathbb{F}_{p^e}$, and suppose that S is not listed in Remark 6.3.6. Then $R_p(S) = d$.*

Remark 6.3.6. The full list of exceptions to Lemma 6.3.5 is:

(i) $S = \mathrm{O}_3^\circ(q) \cong \mathrm{L}_2(q)$ when q is odd, with $R_p(S) = 2$.

(ii) $S = \mathrm{S}_4(2)' \cong A_6$ with $R_2(S) = 3$.

(iii) $S = \mathrm{O}_4^-(q) \cong \mathrm{L}_2(q^2)$, with $R_p(S) = 2$.

(iv) $S = \mathrm{O}_5^\circ(q) \cong \mathrm{S}_4(q)$ when q is odd, with $R_p(S) = 4$.

(v) $S = \mathrm{O}_6^\pm(q) \cong \mathrm{L}_4^\pm(q)$, with $R_p(S) = 4$.

Lemma 6.3.7. *[33, Proposition 5.5.7] Let $S_1, \dots, S_t$ be non-abelian simple groups, and $G = S_1 \times \dots \times S_t$. Then $R_p(G) \geq \sum_{i=1}^{n} R_p(S_i)$.*

Remark 6.3.8. It follows from [18, Chapter 3.7, Theorem 7.2] that we can also apply Lemma 6.3.7 when G is a central product of simple groups.

The following results will be useful when considering $\mathcal{C}_6$-subgroups.

Lemma 6.3.9. *[18, Chapter 5, Theorem 5.5] Let G be an extraspecial group of order r^{2f+1}, where r is prime. Suppose that $\mathbb{F}_{p^e}$ contains a primitive r^2-root of unity, where p is a prime with $p \neq r$. Then the faithful irreducible representations of G over $\mathbb{F}_q$ are all of degree r^f.*

Corollary 6.3.10. *Let G be an extraspecial group of order r^{2f+1}. Suppose that $\mathbb{F}_q$ contains a primitive r^2-root of unity, where p is a prime with $p \neq r$. Then every irreducible character of G is either:*

(i) An unfaithful character of degree 1 with $Z(G)$ contained in the kernel of the character; or

(ii) A faithful character of degree r^f.

Proof. Lemma 6.3.9 classifies all faithful irreducible characters of G, and so it remains to classify the unfaithful irreducible characters. We have that $Z(G) = [G,G] = C_r$, so that the quotient group $G/Z(G)$ is abelian of order r^{2f}; further, since $\Phi(G) = C_r$, it follows that this quotient is elementary abelian, so that $G/Z(G) = C_r^{2f}$. In particular $G/Z(G)$ has r^{2f} linear characters, each of which induces a linear character of G whose kernel contains $Z(G)$. Conversely a nontrivial normal subgroup of $N < G$ must intersect the center of G non-trivially (since N must be a union of conjugacy classes of G, $1 \in N$, $|N| \equiv 0 \mod p$ and the only other elements of G whose conjugacy class has size not divisible by p are elements of the center). Since $Z(G) = C_p$ we have $Z(G) < N$, and hence $Z(G)$ is contained in the kernel of every unfaithful character of G, and so all unfaithful irreducible characters of G must be linear. $\square$

In particular, it follows that the smallest faithful characters of G are irreducible of degree r^f in non-dividing characteristic. When considering $\mathcal{C}_6$-subgroups, in all cases r^f will be the dimension of the classical group Ω, and thus this will rule out containments in $\mathcal{S}^*$-type candidates which are groups of Lie type of smaller dimension, and usually considering $\nu_2(G)$ prevents containments inside the remaining $\mathcal{S}^*$-type candidates.

6.3.2 $\mathrm{SL}_{16}(q)$

Lemma 6.3.11. *Let $\Omega = \mathrm{SL}_{16}(q)$. Then all $\mathcal{S}^*$-maximal subgroups of all almost simple extensions of $\overline{\Omega}$ are maximal.*

Proof. Direct computations show that none of the $\mathcal{S}_1$-candidate subgroups have subgroups of index ≤ 8, nor a subgroup of index 16; hence there are no containments of $\mathcal{S}_1$-candidates in $\mathcal{C}_2$-candidates. For the $\mathcal{S}_2^*$-candidates, [35] gives a lower bound on the degree of a projective representation (and hence a lower bound on the smallest-index proper subgroup), which is larger than 16 for all $\mathcal{S}_2^*$-candidates, so there is no containment of $\mathcal{S}_2^*$-candidates in $\mathcal{C}_2$-candidates.

The only $\mathcal{C}_4$-candidate subgroup is the tensor product $\mathrm{GL}_2(q) \otimes \mathrm{GL}_8(q)$; for a containment here we would require 2- and 8-dimensional irreducible (projective) representations of the $\mathcal{S}^*$-candidate. None of these groups admit a 2-dimensional irreducible representation so there is no possibility of containment here.

The $\mathcal{C}_7$-candidate subgroup is a wreathed tensor of $\mathrm{GL}_4(q)$, so we are interested in groups with a 4-dimensional irreducible projective representation over $\mathbb{F}_q$. The only group which has this is the $\mathcal{S}_2^*$-candidate $2^{\cdot}\mathrm{L}_4(q)$ in characteristic 3. From [41] representations of $\mathrm{L}_4(q)$ in dimension 4 have highest weight $(0, 0, 1)$, and by Lemma 4.1.21 the weights of the tensor product are sums of the weights of the components. In particular, the 10-dimensional $\mathrm{L}_4(q)$-module with weight $(0, 0, 2)$ must appear as a constituent of the wreathed tensor product $\mathrm{GL}_4(q) \otimes \mathrm{GL}_4(q)$, and hence this tensor product cannot contain the irreducible 16-dimensional $\mathcal{S}_2^*$-candidate, whose representation has highest weight $(0, 1, 1)$. Hence there is no containment here.

The only $\mathcal{C}_6$-candidate subgroup is $2^{1+8}.\mathrm{Sp}_8(2)$. Lagrange rules out the possibility of this group containing any of the $\mathcal{S}_1$-candidates, and the only containment of an $\mathcal{S}_2^*$-candidate not ruled out by Lagrange is $2^{\cdot}\mathrm{L}_4(2^2)$. However, this would require a containment of $\mathrm{L}_4(4)$ inside $\mathrm{S}_8(2)$, which is not possible from [8, Table 8.48]. $\square$

Lemma 6.3.12. *Let $\Omega = \mathrm{SL}_{16}(q)$. Then all subgroups which are maximal amongst the geometric subgroups of all almost simple extensions of $\overline{\Omega}$ are maximal.*

Proof. The highest value of ν_q for $\mathcal{S}^*$-candidates is $\nu_q((2, q-1)^{\cdot}\Omega_{10}^{+}(q)) = 20$, and every $\mathcal{C}_1$-candidate has an order with a higher q-part, so there is no possible containment here. We can similarly rule out containments of $\mathcal{C}_3$, $\mathcal{C}_4$ and $\mathcal{C}_8$ in $\mathcal{S}^*$-candidates.

Lemma 6.3.7 tells us that the smallest faithful representation of each of the first three $\mathcal{C}_2$-subgroups has dimension at least 16, so that there are no containments in $\mathcal{S}_2^*$-candidates, and considering the q-part of these candidates rules out containments in $\mathcal{S}_1$-candidates. For the group $(q-1)^{15}.S_{16}$, we see from [57] that A_{16} has no faithful representation in dimension smaller than 14, so that there are no containments of this group in $\mathcal{S}_2^*$-candidates, and $\nu_2(A_{16}) = 14$, which rules out containment in $\mathcal{S}_1$-candidates.

We can rule out containments in $\mathcal{C}_5$ using Lemma 6.3.5 for $\mathcal{S}_2^*$-candidates, and by considering the q_0-part of the orders involved for the $\mathcal{S}_1$-candidates.

Corollary 6.3.10 tells us that the smallest faithful representations of the group 2^{1+8} have dimension 16, ruling out any possible containments in $\mathcal{S}_2^*$-candidates. (Since the $\mathcal{C}_6$-candidate only occurs when $p \equiv 1 \mod 4$ we can always apply the corollary). Otherwise, $\nu_2(\mathrm{Sp}_8(2)) = 16$ which is sufficent to rule out any possible containment in $\mathcal{S}_1$-candidates.

The only $\mathcal{C}_7$-group is $C = (q-1,4).\mathrm{L}_4(q)^2.[(q-1,4)^2].2$. Lagrange or Lemma 6.3.7 rules out any possible containments in all $\mathcal{S}^*$-candidates except $S = \mathrm{HSpin}_{10}^+(q)$. We always have that $|S| < |(2,q-1)^{\cdot}\Omega_{10}^+(q)|$; since we will be ruling out a containment of C inside S considering only the orders of the groups involved, we can take $S = (2,q-1)^{\cdot}\Omega_{10}^+(q)$ to avoid considering multiple cases. We have that

$$|C| = 2(q-1,4)q^{12}(q^4-1)^2(q^3-1)^2(q^2-1)^2,$$
$$|S| = (2,q-1)q^{20}(q^5-1)(q^8-1)(q^6-1)(q^4-1)(q^2-1).$$

When q is odd, by Lemma 6.3.1, we see that $\nu_2(C) = 4r_\pm + 2r_- + \min\{2, r_-\} + 7$, where $r_- = \nu_2(q-1)$ and $r_\pm = \max\{\nu_2(q-1), \nu_2(q+1)\}$, whereas $\nu_2(S) = 4r_\pm + r_- + 8$. Since $r_- \geq 1$ it follows that $\nu_2(C) > \nu_2(S)$ and so there is no containment here.

When q is even, the ratio of orders (after some simplification) is

$$\frac{|S|}{|C|} = \frac{q^8(q^5-1)(q^4+1)(q^3+1)}{(q^3-1)(q^2-1)}.$$

By Lemma 6.3.3 there exists a prime l such that $q^3 \equiv 1 \mod l$ but $q^2 \not\equiv 1 \mod l$ and $q \not\equiv 1 \mod l$. The integer l does not divide q^8 as q is even. If l divides q^5-1 then $q^5 \equiv 1 \mod l$ and $q^3 \equiv 1 \mod l$, so $q^2 \equiv 1 \mod l$ which is not true. If l divides q^4+1 then $q^4 \equiv -1 \mod l$ and $q^5 \equiv 1 \mod l$, so $q \equiv -1 \mod l$ and $l|(q+1)$, so l divides q^2-1, which is not possible. If l divides q^3+1 then $q^2 \equiv -1 \mod l$ and $q^3 \equiv 1 \mod l$ so $q \equiv -1 \mod l$, which is again not possible. Hence a suitable power of l divides $|C|$ but not $|S|$, so by Lagrange there can be no containment between these groups. □

6.3.3 $\mathrm{SU}_{16}(q)$

Lemma 6.3.13. *Let $\Omega = \mathrm{SU}_{16}(q)$. Then all $\mathcal{S}^*$-maximal subgroups of all almost simple extensions of $\overline{\Omega}$ are maximal.*

Proof. We eliminate containments in $\mathcal{C}_2$- and $\mathcal{C}_4$-candidates via similar methods to those used in Lemma 6.3.11.

For $\mathcal{C}_7$, we see from [32] that in characteristic 5 the group $4_2^{\cdot}\mathrm{L}_3(4).2_2$ has two weakly-equivalent nontrivial 4-dimensional faithful characters, but these are the only 4-dimensional projective characters and inspection (for example on elements of class 4A) shows that the 16-dimensional character is not a tensor product of two of these characters. We see by inspection of the remaining groups, or from [8, Table 4.4] that there are no other $\mathcal{S}_1$-candidates with a 4-dimensional nontrivial projective representation.

In groups in Class $\mathcal{C}_6$, Lagrange allows the possibility of a containment of $4_2^{\cdot}\mathrm{L}_3(4).2_2$ inside $\mathrm{Sp}_8(2)$, but there is no faithful representation of $4_2^{\cdot}\mathrm{L}_3(4)$ in characteristic 2, and we can again rule out a containment of $\mathrm{SU}_4(q).(q+1,4)$ inside $\mathrm{Sp}_8(2)$ by [8, Table 8.48]. $\square$

Lemma 6.3.14. *Let $\Omega = \mathrm{SU}_{16}(q)$. Then all subgroups which are maximal amongst the geometric subgroups of all almost simple extensions of $\overline{\Omega}$ are maximal.*

Proof. The proofs here are very similar to those in Lemma 6.3.12, so we only sketch the details.

Considering q-parts rules out containments in $\mathcal{C}_1$ and $\mathcal{C}_4$. Lemma 6.3.7 rules out containments for the first three $\mathcal{C}_2$-candidates, and containments for the last two candidates can be ruled out by [57] and considering q-parts respectively. Lemma 6.3.5 rules out containments involving $\mathcal{C}_5$-candidates, and Lagrange and Corollary 6.3.10 deals with containments of the $\mathcal{C}_6$-canddate in $\mathcal{S}_1$- and $\mathcal{S}_2^*$-candidates respectively.

Again similarly to Lemma 6.3.12, the only containment of $\mathcal{C}_7$-candidates inside $\mathcal{S}^*$-subgroups that cannot be immediately ruled out by Lagrange or minimal degrees of representations is the containment $C < S$ where C denotes the $\mathcal{C}_7$ group $(q+1,4).\mathrm{U}_4(q)^2.(q+1,4)^2.2$ and S is an extension of $\mathrm{O}_{10}^-(q)$. As in the linear case we take $S = (2,q-1)^{\cdot}\Omega_{10}^-(q)$. We have

$$|C| = 2(4,q-1)q^{12}(q^4-1)^2(q^3+1)^2(q^2-1)^2,$$
$$|S| = (2,q-1)q^{20}(q^5+1)(q^8-1)(q^6-1)(q^4-1)(q^2-1).$$

When q is odd we can count 2-parts and conclude that $\nu_2(C) > \nu_2(S)$; when q is even the ratio of orders is

$$\frac{|S|}{|C|} = \frac{q^8(q^5+1)(q^4+1)(q^3-1)}{2(q^2-1)(q^3+1)}$$

and we can apply Lemma 6.3.3 to $(q^6-1) = (q^3-1)(q^3+1)$ to find a prime l such that l divides q^3+1 but not q^i-1 for $i < 6$. Similarly to before l cannot divide any of the terms in the numerator, so there is no possible containment $C < S$.

□

6.3.4 $\mathrm{Sp}_{16}(q)$

Lemma 6.3.15. *Let $\Omega = \mathrm{Sp}_{16}(q)$. Then all $\mathcal{S}^*$-maximal subgroups of all almost simple extensions of $\overline{\Omega}$ are maximal.*

Proof. As before, we only need to consider containments in Class $\mathcal{C}_i$ for $i = 2, 4, 6, 7$. Note that there are no $\mathcal{C}_7$-candidate subgroups of $\mathrm{Sp}_{16}(q)$.

We first consider the $\mathcal{C}_2$ cases. From the discussion in Section 6.3.1, to be contained in a group of the form $\hat{G} \wr S_t$ we require the $\mathcal{S}^*$-candidate to have a subgroup of index t. From Table 7.5 we see that $t \leq 8$. From direct computations for $\mathcal{S}_1$-candidates and [35] for $\mathcal{S}_2^*$-candidates, the only $\mathcal{S}^*$-candidate with a subgroup of index 8 or fewer is $2^{\cdot}A_8$, which has an index 8 subgroup isomorphic to $2^{\cdot}A_7$, and the only character of $2^{\cdot}A_7$ of degree 2 is the direct sum of two copies of the trivial representation, which induces up to the permutation representation of $2^{\cdot}A_8$, whose image is isomorphic to A_8. Hence the 16-dimensional faithful irreducible representation of $2^{\cdot}A_8$ is primitive. It remains to consider containment in the $\mathcal{C}_2$-subgroup $\mathrm{GL}_8(q).2$. However, if S is an $\mathcal{S}^*$-candidate and we have a containment $S < \mathrm{GL}_8(q).2 < \mathrm{Sp}_{16}(q)$ then by construction of the $\mathcal{C}_2$-subgroup, S has an index 2 subgroup which is reducible, and this is not true for any of the $\mathcal{S}^*$-candidates.

For the groups in Class $\mathcal{C}_4$, the candidate groups are $\mathrm{Sp}_4(q) \otimes \Omega_4^{\pm}(q)$ or $\mathrm{Sp}_2(q) \otimes \Omega_8^{\pm}(q)$, and in order for an $\mathcal{S}^*$-candidate subgroup to be contained in one of these, it must have a nontrivial (projective) representation contained in each component of the tensor product. We can look at the character tables of the $\mathcal{S}_1$-candidates (or [56] in the case of A_{18}) to see that none of these have representations in degree 2 or 4. For the $\mathcal{S}_2^*$-candidates, it is clear that $\mathrm{Sp}_4(q)$ does not have a representation with image contained in $\mathrm{Sp}_2(q)$ or $\Omega_4^{\pm}(q)$, and although $\mathrm{SL}_2(q)$ has representations in these dimensions, all irreducible even-dimensional representations preserve a symplectic form by [8, Proposition 5.3.6], so we have no representations contained in $\Omega_t^{\pm}(q)$ with $t = 4, 8$. Hence there is no containment of $\mathcal{S}^*$-candidates in $\mathcal{C}_4$-candidates.

The only $\mathcal{C}_6$-candidate subgroups are extensions by $\Omega_8^-(2)$, and containment of an $\mathcal{S}^*$-candidate subgroup in the $\mathcal{C}_6$-candidate would require containment of the nonabelian simple composition factor of the $\mathcal{S}^*$-candidate in $\Omega_8^-(2)$. Lagrange rules out all containments except the $\mathcal{S}_1$-candidate $2^{\cdot}A_8$ and the $\mathcal{S}_2^*$-candidate $\mathrm{Sp}_4(3)$, and a direct check shows that there are subgroups of $\Omega_8^-(2)$ isomorphic to both A_8 and $\mathrm{S}_4(3)$. The former occurs only when $q = 7$, and the latter when $q = 3$, so we can perform direct computer computations. There is no containment of $2^{\cdot}A_8$ in this geometric-type subgroup, since from [32] matrices of order 5 in the 16-dimensional representation have trace 1, whereas from computations in `geomcont` we see that elements of order 5 in $2^{1+8}.\mathrm{SO}_8^-(2)$ have trace 3. Similar computations, also in `geomcont`, show that the traces of elements of order 5 also differ between the geometric-type candidate and the $\mathcal{S}_2^*$-type candidate $\mathrm{Sp}_4(3)$, ruling out containments in this case as well. □

Lemma 6.3.16. *Let $\Omega = \mathrm{Sp}_{16}(q)$. Then all subgroups which are maximal amongst the geometric subgroups of all almost simple extensions of $\overline{\Omega}$ are maximal.*

Proof. The highest value of ν_p appearing in any $\mathcal{S}^*$-type candidate is $\nu_2(A_{18}) = 15$, or $\nu_q(\mathrm{Sp}_4(q)) = 4$. Checking $\nu_p(G)$, for Ω a classical group in characteristic p and G a geometric-type subgroup, we can eliminate possible containments of all $\mathcal{C}_1$, $\mathcal{C}_3$ and $\mathcal{C}_8$ candidates, and also eliminates all $\mathcal{C}_2$-candidates except $\mathrm{Sp}_2(q)^8 : S_8$, which we can rule out by Lemma 6.3.5 and Lemma 6.3.7.

Considering q-parts also rules out possible containments in $\mathcal{C}_4$-candidates. For the $\mathcal{C}_5$-candidates, Lemma 6.3.5 rules out possible containments in all $\mathcal{S}_2^*$-candidates, and Lagrange rules out containments in $\mathcal{S}_1$-candidates.

Finally, we consider the $\mathcal{C}_6$-candidates. The order of these candidates is divisible by 2^{21}, which rules out containments in $\mathcal{S}_1$-candidates. The group $\Omega_8^-(2)$ has no faithful 2- or 4-dimensional representation in any characteristic so there is no containment in the $\mathcal{S}_2^*$-candidates $\mathrm{SL}_2(q)$ or $\mathrm{Sp}_4(q)$. □

6.3.5 $\Omega_{16}^+(q)$

Lemma 6.3.17. *Let $\Omega = \Omega_{16}^+(q)$. Then all $\mathcal{S}^*$-maximal subgroups of all almost simple extensions of $\overline{\Omega}$ are maximal.*

Proof. We first consider $\mathcal{C}_2$-candidates. It is straightforward to check that none of the nonabelian composition factors of the $\mathcal{S}_1$-candidates have subgroups of index

2, 4, 8 or 16, and as in Lemma 6.2.6 there are no $\mathcal{S}_2^*$-candidates with permutation representations on fewer than 17 points. Hence there are no abstract containments of any $\mathcal{S}^*$-candidates in $\mathcal{C}_2$-candidates.

For the $\mathcal{C}_4$-candidates, we can see from [8, Theorem 4.3,3] that there are no $\mathcal{S}_1$-candidates with 2- or 4-dimensional faithful irreducible projective representations, so there is no containment of any of the $\mathcal{S}_1$-candidates in $\mathcal{C}_4$-candidates. None of the $\mathcal{S}_2^*$-candidates have 2-dimensional representations over $\mathbb{F}_q$, so the only possibility that remains is containment of an $\mathcal{S}_2^*$-candidate subgroup in the $\mathcal{C}_4$-candidate $\mathrm{SO}_4^+(q) \otimes \mathrm{SO}_4^-(q)$. However, since $\mathrm{O}_4^+(q) \cong \mathrm{L}_2(q) \times \mathrm{L}_2(q)$, a group with an irreducible representation inside $\mathrm{O}_4^+(q)$ would also need to have its nonabelian simple composition factor appear as a subgroup of $\mathrm{L}_2(q)$, which none of the $\mathcal{S}_2^*$-candidates do; hence there are no containments inside $\mathcal{C}_4$-candidates.

In a similar way we can rule out the possibility of containment in all of the $\mathcal{C}_7$-subgroups except the tensor wreath product $\mathrm{Sp}_4(q) \wr S_2$. Since the nonabelian composition factor of the $\mathcal{S}_2$-candidate must be contained in $\mathrm{S}_4(q)$ (and $\mathrm{Sp}_4(q) \wr S_2$ is not maximal when q is even), we can rule out all containments except $\mathrm{L}_2(q^4).4$, and it follows from [8, Table 8.12] that there is no containment of $\mathrm{L}_2(q^4)$ in $\mathrm{S}_4(q)$; hence no containments in $\mathcal{C}_7$-groups are possible.

The $\mathcal{C}_6$-candidate subgroups of Ω have nonabelian composition factor $\mathrm{O}_8^+(2)$. Lagrange eliminates the possibility of all containments between $\mathcal{S}_1$-candidates except $(2^{\cdot})A_{10}$. In characteristic 2 the group A_{10} has a single representation in dimension 8, which preserves a symplectic form and hence does not embed into $\mathrm{O}_8^+(2)$. For the $\mathcal{S}_2^*$-candidates Lagrange rules out any possible containments inside $\mathrm{O}_8^+(2)$. $\square$

Lemma 6.3.18. *Let $\Omega = \Omega_{16}^+(q)$. Then all subgroups which are maximal amongst the geometric subgroups of all almost simple extensions of $\overline{\Omega}$ are maximal.*

Proof. The highest value of ν_q on $\mathcal{S}^*$-candidates is $\nu_q(2^{\cdot}\Omega_9^\circ(q)) = q^{16}$. Thus by considering the q-part of geometric type candidates, we can rule out containment in all $\mathcal{C}_1$- and $\mathcal{C}_3$-candidates.

For the $\mathcal{C}_2$-candidates, we can use Lemma 6.3.7 and [57] to rule out containments in $\mathcal{S}_2^*$-candidates for all except $G = \Omega_4^-(q)^4.2^{3(2,q-1)}.S_4$, as $\mathrm{O}_4^-(q) \cong \mathrm{L}_2(q^2)$. In this case, the group has order $\frac{q^8(q^2+1)^4(q^2-1)^4}{(q-1,2)^4}2^{3(q-1,2)}2^3 3$. The q-part rules out possible containments in all $\mathcal{S}_2^*$-candidates except $\mathrm{Sp}_4(q^2).2$ and $S = (2,q-1)^{\cdot}\Omega_9^\circ(q)$. The order of the former divides the order of the latter, which is

$$(2, q-1)q^{16}(q^8-1)(q^6-1)(q^4-1)(q^2-1).$$

When q is odd, set $r = \max\{\nu_2(q-1), \nu_2(q+1)\}$; then using Corollary 6.3.1 and

Corollary 6.3.2, $\nu_2(G) = 4r + 13$ and $\nu_2(S) = 4r + 8$, so there is no possible containment here when q is odd. When q is even, we can construct the ratio

$$\frac{|S|}{|G|} = \frac{q^8(q^4+1)(q^6-1)}{(q^2+1)^2(q-1)2^6 3}.$$

By Lemma 6.3.3 there is a prime l which divides $q^4 - 1$ but not $q^i - 1$ for $i = 1, 2, 3$. Since $q^4 - 1 = (q^2 + 1)(q^2 - 1)$ and l does not divide $q^2 - 1$, l divides $q^2 + 1$, and we can show that l cannot divide any term in the numerator. Indeed, if l divides $q^6 - 1$ then $q^6 \equiv q^4 \equiv 1 \mod l$, so that $q^2 \equiv 1 \mod l$ which contradicts choice of l. Similarly if l divides $q^4 + 1$ then since l divides $q^4 - 1$ we have that l divides 2, which is also impossible. Hence this ratio is not an integer, and so there is no containment when q is even either.

For containments of $\mathcal{C}_2$-candidates inside $\mathcal{S}_1$-candidates, considering the 2-parts rules out containments involving $2^{15}.A_{16}$ (or its extensions) and considering q-parts rules out the extensions of $\Omega_8^{\pm}(q)^2$. The extensions of $\Omega_4^{\pm}(q)$ have q-part q^8, which rules out all possible containments when q is odd. Counting 2-parts when q is even rules out all containments except that of $\Omega_4^{\pm}(2).2^3.S_4$ inside A_{17}, and the order of the former does not divide the order of the latter. The 2-part of the groups $\Omega_2^{\pm}(q)^8.2^{7(2,q-1)}.S_8$ is at least 14, leaving containment in A_{17} as the only possibility. Since A_{17} also has 2-part 14, and no other prime power larger than 6 dividing it, the only possibility for containment is when q is even and $\Omega_2^{\pm}(q)$ is trivial; i.e. when the group is an extension of $\Omega_2^{+}(2)$, and in this case the $\mathcal{C}_2$-group is already not maximal.

We can rule out containments of $\mathcal{C}_5$ using Lemma 6.3.5 for $\mathcal{S}_2^*$-candidates and by considering q_0-parts for $\mathcal{S}_1$-candidates. For $\mathcal{C}_6$-candidates we can rule out containment in $\mathcal{S}_2^*$-candidates using Corollary 6.3.10, and in $\mathcal{S}_1$-candidates by considering 2-parts.

We next consider the $\mathcal{C}_4$-candidates. The group $(\mathrm{Sp}_2(q)\circ\mathrm{Sp}_8(q)).(2, q-1)$ has q-part 17, ruling out all possible containments in $\mathcal{S}^*$-candidates. We next consider the group $C = (\mathrm{SO}_4^+(q) \circ \mathrm{SO}_4^-(q)).2^2$. We can use Lemma 6.3.5, Lemma 6.3.7 and Remark 6.3.8 to conclude that the smallest dimension of a representation of C is at least 6, leaving the only possible containment in $\mathcal{S}_2^*$-candidates as inside the group $S = 2^{\cdot}\mathrm{SO}_9^{\circ}(q)$. Since $\mathrm{O}_4^-(q) \cong \mathrm{L}_2(q^2)$, it follows that the smallest representation of $\mathrm{SO}_4^-(q)$ over $\mathbb{F}_q$ is the natural 4-dimensional representation; hence from [8, Table 8.58], the only maximal subgroups of C which could contain a group isomorphic to S (or the quotient of S by its centre) are the groups $\Omega_8^{\pm}(q).2$. (Since C is not itself a direct product it cannot be contained in the direct product of two groups of smaller dimension, and it also cannot be written over a smaller field). If we had such a containment, it would follow that there was also a containment of $\mathrm{SO}_4^+(q)\circ\mathrm{SO}_4^-(q) < \Omega_8^{\pm}(q)$. However, considered as subgroups of $\Omega_{16}^+(q)$, both S and C have irreducible representations; hence if there is a containment $C < \Omega_8^{\pm}(q) < S < \Omega_{16}^+(q)$ then we must have an irreducible 16-dimensional representation of $\Omega_8^{\pm}(q)$ by Lemma 4.5.2, which is not possible by [41]. Thus there are no containments of $\mathcal{C}_4$-groups inside $\mathcal{S}_2^*$-candidates. For the $\mathcal{S}_1$-candidates, since $p \neq 2$ and the q-part of C is 4, the only possibility for containment is when $q = 3$. Lagrange rules out all possible containments except $C < A_{17}$, which can be ruled out by a direct calculation.

For the $\mathcal{C}_7$-candidates, we can use Lemma 6.3.7 to leave the only possible containments inside $\mathcal{S}_2^*$-candidates as any of the candidates inside $(2, q-1)^{\cdot}\Omega_9^{\circ}(q)$, and $2.\mathrm{PSO}_4^-(q)^2.2^3$ inside $\mathrm{Sp}_4(q^2).2$. Similarly to the previous paragraph, containments of $\mathrm{S}_2(q)^4$ or $\mathrm{S}_4(q)^2$ inside $(2, q-1)^{\cdot}\Omega_9^{\circ}(q)$ are impossible by considering its maximal subgroups. From [8, Table 8.12] we can rule out containments of $2.\mathrm{PSO}_4^-(q)^2.2^3 \cong 2.\mathrm{L}_2(q^2).2^3$ inside all maximal subgroups of $\mathrm{Sp}_4(q^2)$ by considering either dimension or order. □

6.3.6 $\Omega_{16}^-(q)$

In this section let $\Omega = \Omega_{16}^-(q)$. Recall that there are no $\mathcal{S}_2^*$-candidate subgroups of Ω, and that the $\mathcal{S}_1$-candidates are A_{18} with $q = p = 3$, A_{17} with $q = p \neq 3$ and p not a square modulo 17, and two representations of $\mathrm{L}_2(17)$; $\mathrm{L}_2(17)_1$ with $q = p \neq 3$ and not a square modulo 17, and $\mathrm{L}_2(17)_2$ with p not a square modulo 17, and $q = p$ if $p \equiv \pm 1 \mod 9$ and $q = p^3$ otherwise.

Note that there are no geometric-type candidates in Classes $\mathcal{C}_i$ for $i = 2, 4, 6, 7$, so there are no possible containments of $\mathcal{S}^*$-candidates in geometric-type candidates.

Lemma 6.3.19. *Let $\Omega = \Omega_{16}^-(q)$. Then all subgroups which are maximal amongst the geometric subgroups of all almost simple extensions of $\overline{\Omega}$ are maximal.*

Proof. The highest q-parts of $\mathcal{S}^*$-subgroups are $\nu_2(A_{18}) = 15$ and $\nu_3(A_{17}) = 8$. The q-parts of all $\mathcal{C}_1$- and $\mathcal{C}_3$-candidates are larger than 15 in all cases, as is the q_0-part of the $\mathcal{C}_5$-candidate. □

6.3.7 $\Omega_{17}^{\circ}(q)$

Lemma 6.3.20. *Let $\Omega = \Omega_{17}^{\circ}(q)$. Then all $\mathcal{S}^*$-maximal subgroups of all almost simple extensions of $\overline{\Omega}$ are maximal.*

Proof. Recall that the definition of Class $\mathcal{S}^*$ means that the only geometric-type subgroups we need to consider in this section are those in Classes $\mathcal{C}_i$ for $i = 2, 4, 6, 7$. From Table 7.13 we see that the only such geometric-type subgroup of Ω is $\mathrm{O}_1^\circ(q) \wr S_{17}$ or $\mathrm{O}_1^\circ(q) \wr A_{17}$, which consists of generalised permutation matrices whose entries take the values ± 1. (For notational convenience we will refer to the group as $\mathrm{O}_1^\circ(q) \wr S_{17}$ throughout, but the arguments also work when the group is $\mathrm{O}_1^\circ(q) \wr A_{17}$). This acts imprimitively on the 17-dimensional module V of $\Omega_{17}^\circ(q)$, where we can write the blocks of V as 1-dimensional subspaces spanned by the basis elements of V. Conversely, any matrix group which acts imprimitively on V with blocks of size 1 must consist of generalised permutation matrices.

Lagrange rules out possible containments (in either direction) between $A_{18}.2$ and $\mathrm{O}_1^\circ(q) \wr S_{17}$. We are thus left with the possiblity of containments of (extensions of) $\mathrm{L}_2(16)$ in $\mathrm{O}_1^\circ(q) \wr S_{17}$. From [8, Section 8.2, Table 8.1] we see that $\mathrm{L}_2(16)$ has a subgroup H of index 17 isomorphic to $2^4 : 15$. Computer calculations in `geomcont` show that each 17-dimensional character of $\mathrm{L}_2(16)$ is induced from a different 1-dimensional character of H.

For $G = \mathrm{L}_2(16)_1$, the calculations show that the character ring of the 1-dimensional character of H inducing the representation is generated by z_3, whilst the character ring of G involves no irrationalities. Hence when $p \equiv 1 \mod 4$, the character ring of the representation of H is $\mathbb{F}_{p^2}$ whereas the character ring of the representation of G is $\mathbb{F}_p$. Thus when $p \equiv 1 \mod 4$, the representation of G over $\mathbb{F}_p$ is primitive. When $p \equiv 3 \mod 4$ both representations are realisable over $\mathbb{F}_p$, so the representation is imprimitive in this case. In order to have a containment of G inside $\mathrm{O}_1^\circ(q) \wr S_{17}$, we require the stabiliser of a 1-dimensional block of G to be isomorphic to a subgroup of $\mathrm{O}_1^\circ(q)$, which consists of the scalars ± 1. The action of H on this 1-dimensional vector space is as the 1-dimensional representation of H. Again from the computations, we see that the character value of the 1-dimensional representation of H on elements of order 3 is z_3 or $-1-z_3$; hence elements of order 3 act non-trivially on the vector space and so the stabiliser does not act as a subgroup of $\mathrm{O}_1^\circ(q)$. Hence in no case is there a containment of $\mathrm{L}_2(16)_1$ in $\mathrm{O}_1^\circ(q) \wr S_{17}$.

The computations for $\mathrm{L}_2(16)_2$ and $\mathrm{L}_2(16)_3$ are similar. The 1-dimensional representation inducing $\mathrm{L}_2(16)_2$ takes character value z_5 on elements of order 5, so when the representation is imprimitive elements of H of order 5 act nontrivially on vectors in the blocks of the action (and there can obviously be no containment when the representation is primitive); hence there is no containment here. A similar argument holds for $\mathrm{L}_2(16)_3$. □

Lemma 6.3.21. *Let $\Omega = \Omega_{17}^{\circ}(q)$. Then all subgroups which are maximal amongst the geometric subgroups of all almost simple extensions of $\overline{\Omega}$ are maximal.*

Proof. Considering 2-parts rules out the possibility of containment of $\mathcal{C}_2$-candidates in any $\mathcal{S}^*$-candidates, and considering q-parts and q_0-parts for $\mathcal{C}_1$-candidates and $\mathcal{C}_5$ candidates respectively rules out containments of these inside any $\mathcal{S}^*$-candidate.

$\square$

Chapter 7

Results

In this chapter, we summarise the results of the previous chapters and provide the tables as referred to in Theorem 1.1.1. We present the tables in the same format as [8, Chapter 8], and refer the reader there for more details on the information contained in these tables.

For each classical group Ω, we provide two tables (except for $\mathrm{SL}^{\pm}_{17}(q)$ where there are no $\mathcal{S}^*$-maximal candidates); one describing the geometric-type maximal subgroups of Ω and one describing the maximal subgroups of Ω in class $\mathcal{S}$. We describe the columns in the tables below:

- 'Class' only occurs in the geometric-type tables, and denotes the number i corresponding to the Aschbacher class $\mathcal{C}_i$ that the group falls into.
- 'Group' denotes the isomorphism type of the group. The isomorphism type for $\mathcal{S}^*$-maximal subgroups includes scalar matrices in Ω.
- 'Notes' denotes any restrictions on q required for the listed group to be maximal, and also describes any novelties.
- 'Classes' gives the number of conjugacy classes. This is denoted c, and may also appear in the stabiliser or the structure of the group.
- 'Stabiliser' provides the class stabiliser of the group.

Results indicated by an asterisk denote conjectures, and references are given to where the conjecture is discussed in more detail.

The proof of correctness of the geometric-type tables are from [33, Chapter 3] with the exact structures of some of these groups coming from the computations in [8, Chapter 2]. The proof of correctness of the groups in class $\mathcal{S}$ come from Chapters 3 and 4, and the proof of maximality of both of these tables comes from Chapter 6.

7.1 $\mathrm{SL}_{16}(q)$

Table 7.1: The maximal subgroups of $\mathrm{SL}_{16}(q)$ of geometric type
$d := |Z(\mathrm{SL}_{16}(q))| = (q-1,16), |\delta| = d, |\phi| = e, |\gamma| = 2, q = p^e$

Class	Group	Notes	Classes	Stabiliser
1	$E_q^{15} : \mathrm{GL}_{15}(q)$		2	$\langle\delta,\phi\rangle$
1	$E_q^{28} : (\mathrm{SL}_{14}(q) \times \mathrm{SL}_2(q)) : (q-1)$		2	$\langle\delta,\phi\rangle$
1	$E_q^{39} : (\mathrm{SL}_{13}(q) \times \mathrm{SL}_3(q)) : (q-1)$		2	$\langle\delta,\phi\rangle$
1	$E_q^{48} : (\mathrm{SL}_{12}(q) \times \mathrm{SL}_4(q)) : (q-1)$		2	$\langle\delta,\phi\rangle$
1	$E_q^{55} : (\mathrm{SL}_{11}(q) \times \mathrm{SL}_5(q)) : (q-1)$		2	$\langle\delta,\phi\rangle$
1	$E_q^{60} : (\mathrm{SL}_{10}(q) \times \mathrm{SL}_6(q)) : (q-1)$		2	$\langle\delta,\phi\rangle$
1	$E_q^{63} : (\mathrm{SL}_9(q) \times \mathrm{SL}_7(q)) : (q-1)$		2	$\langle\delta,\phi\rangle$
1	$E_q^{64} : (\mathrm{SL}_8(q) \times \mathrm{SL}_8(q)) : (q-1)$		1	$\langle\delta,\phi,\gamma\rangle$
1	$\mathrm{GL}_{15}(q)$	N1	1	$\langle\delta,\phi,\gamma\rangle$
1	$(\mathrm{SL}_2(q) \times \mathrm{SL}_{14}(q)) : (q-1)$	N1	1	$\langle\delta,\phi,\gamma\rangle$
1	$(\mathrm{SL}_3(q) \times \mathrm{SL}_{13}(q)) : (q-1)$	N1	1	$\langle\delta,\phi,\gamma\rangle$
1	$(\mathrm{SL}_4(q) \times \mathrm{SL}_{12}(q)) : (q-1)$	N1	1	$\langle\delta,\phi,\gamma\rangle$
1	$(\mathrm{SL}_5(q) \times \mathrm{SL}_{11}(q)) : (q-1)$	N1	1	$\langle\delta,\phi,\gamma\rangle$
1	$(\mathrm{SL}_6(q) \times \mathrm{SL}_{10}(q)) : (q-1)$	N1	1	$\langle\delta,\phi,\gamma\rangle$
1	$(\mathrm{SL}_7(q) \times \mathrm{SL}_9(q)) : (q-1)$	N1	1	$\langle\delta,\phi,\gamma\rangle$
1	$E_q^{29} : \mathrm{SL}_{14}(q) : (q-1)^2$	N1	1	$\langle\delta,\phi,\gamma\rangle$
1	$E_q^{52} : (\mathrm{SL}_2(q)^2 \times \mathrm{SL}_{12}(q)) : (q-1)^2$	N1	1	$\langle\delta,\phi,\gamma\rangle$
1	$E_q^{69} : (\mathrm{SL}_3(q)^2 \times \mathrm{SL}_{10}(q)) : (q-1)^2$	N1	1	$\langle\delta,\phi,\gamma\rangle$
1	$E_q^{80} : (\mathrm{SL}_4(q)^2 \times \mathrm{SL}_8(q)) : (q-1)^2$	N1	1	$\langle\delta,\phi,\gamma\rangle$
1	$E_q^{85} : (\mathrm{SL}_5(q)^2 \times \mathrm{SL}_6(q)) : (q-1)^2$	N1	1	$\langle\delta,\phi,\gamma\rangle$
1	$E_q^{84} : (\mathrm{SL}_6(q)^2 \times \mathrm{SL}_4(q)) : (q-1)^2$	N1	1	$\langle\delta,\phi,\gamma\rangle$
1	$E_q^{77} : (\mathrm{SL}_7(q)^2 \times \mathrm{SL}_2(q)) : (q-1)^2$	N1	1	$\langle\delta,\phi,\gamma\rangle$
2	$\mathrm{SL}_8(q)^2.(q-1).2$		1	$\langle\delta,\phi,\gamma\rangle$
2	$\mathrm{SL}_4(q)^4.(q-1)^3.S_4$		1	$\langle\delta,\phi,\gamma\rangle$
2	$\mathrm{SL}_2(q)^8.(q-1)^7.S_8$	$q \neq 2$	1	$\langle\delta,\phi,\gamma\rangle$
2	$(q-1)^{15}.S_{16}$	$q \geq 5$	1	$\langle\delta,\phi,\gamma\rangle$
3	$\left((q-1,8)(q+1) \circ \mathrm{SL}_8(q^2)\right) \frac{(q^2-1,8)}{(q-1,8)}.2$		1	$\langle\delta,\phi,\gamma\rangle$
4	$(\mathrm{SL}_2(q) \circ \mathrm{SL}_8(q)).(q-1,2)^2$	$q \neq 2$	$(q-1,2)$	$\langle\delta^c,\phi,\gamma\rangle$
5	$\mathrm{SL}_{16}(q_0).c$	$q = q_0^r$, r prime	$\left(\frac{q-1}{q_0-1},16\right)$	$\langle\delta^c,\phi,\gamma\rangle$
6	$((q-1,16) \circ 2^{1+8}).\mathrm{Sp}_8(2)$	$q = p \equiv 1 \mod 4$	$(q-1,16)$	$\langle\delta^c,\phi,\gamma\rangle$
7	$(q-1,4).\mathrm{L}_4(q)^2.[(q-1,4)^2].2$		$(q-1,4)$	$\langle\delta^c,\phi,\gamma\rangle$
8	$(q-1,16)^{\cdot}\mathrm{S}_{16}(q).k$		$(q-1,8)$	$\langle\delta^c,\phi,\gamma\rangle$
8	$\mathrm{SO}_{16}^+(q).[(q-1,16)]$	q odd	$\frac{(q-1,16)}{2}$	$\langle\delta^c,\phi,\gamma\rangle$
8	$\mathrm{SO}_{16}^-(q).[(q-1,16)]$	q odd	$\frac{(q-1,16)}{2}$	$\langle\delta^c,\phi\delta^{(p-1)/2},\gamma\delta^{-1}\rangle$
8	$\mathrm{SU}_{16}(q_0).[(q_0-1,16)]$	$q = q_0^2$	$(q_0-1,16)$	$\langle\delta^c,\phi,\gamma\rangle$

$$k = \begin{cases} 1 & \text{if } q \equiv 1 \mod 16 \text{ or } q \text{ is even,} \\ 2 & \text{otherwise.} \end{cases}$$

N1 - Maximal under subgroups not contained in $\langle\delta,\phi\rangle$

Table 7.2: The maximal subgroups of $\mathrm{SL}_{16}(q)$ in class $\mathcal{S}$

$d := |Z(\mathrm{SL}_{16}(q))| = (q-1, 16), |\delta| = d, |\phi| = e, |\gamma| = 2, q = p^e$

Group	Notes	Classes	Stabiliser
$2^{\cdot}A_{12}$	$q = 3$	2	$\langle\gamma\rangle$
$d \circ 2^{\cdot}A_{11}$	$q = p \equiv 1, 3, 4, 5, 9 \mod 11$, $p \neq 3$	d	$\langle\gamma\rangle$
$d \times M_{12}$	$q = p \equiv 1, 25, 31, 37, 49 \mod 66$	d	$\langle\gamma\rangle$
$d \times M_{12}$	$q = p \equiv 5, 23, 47, 53, 59 \mod 66$	d	$\langle\gamma\delta\rangle$
$d \circ \mathrm{SL}_2(31)$	$q = p$ square mod 31, $p \equiv \pm 1 \mod 8$	d	$\langle\gamma\rangle$
$d \circ \mathrm{SL}_2(31)$	$q = p$ square mod 31, $p \equiv \pm 3 \mod 8$	d	$\langle\gamma\delta\rangle$
$d \times \mathrm{L}_3(3)$	$q = p \equiv 1, 55, 61 \mod 78$	d	$\langle\gamma\rangle$
$d \times \mathrm{L}_3(3)$	$q = p \equiv 1, 55, 61 \mod 78$	d	$\langle\gamma\rangle$
$d \times \mathrm{L}_3(3)$	$q = p \equiv 29, 35, 53 \mod 78$	d	$\langle\gamma\delta\rangle$
$d \times \mathrm{L}_3(3)$	$q = p \equiv 29, 35, 53 \mod 78$	d	$\langle\gamma\delta\rangle$
$\mathrm{SL}_4(q).2$	$q = 3^e$, e odd	2	$\langle\phi, \gamma\rangle$
$8 \circ \mathrm{SL}_4(q).2$	$q = 3^e$, $e \equiv 2 \mod 4$	4	$\langle\delta^4, \phi, \gamma\rangle$
$16 \circ \mathrm{SL}_4(q)$	$q = 3^e$, $e \equiv 0 \mod 4$	4	$\langle\delta^4, \phi, \gamma\rangle$
$\mathrm{L}_4(q^2).2$	q even	1	$\langle\phi, \gamma\rangle$
$2 \times (\mathrm{L}_4(q^2).(4 \times 2))$	$q \equiv 3 \mod 4$	2	$\langle\phi, \gamma\rangle$
$4 \circ 2^{\cdot}\mathrm{L}_4(q^2).(4 \times 2)$	$q \equiv 5 \mod 8$	4	$\langle\phi, \gamma\rangle$
$8 \circ 2^{\cdot}\mathrm{L}_4(q^2).2^2$	$q \equiv 9 \mod 16$	4	$\langle\delta^4, \phi, \gamma\rangle$
$16 \circ 2^{\cdot}\mathrm{L}_4(q^2).2$	$q \equiv 1 \mod 16$	4	$\langle\delta^4, \phi, \gamma\rangle$
$\Omega^+_{10}(q)$	q even	1	$\langle\phi, \gamma\rangle$
$2^{\cdot}\Omega^+_{10}(q).2$	$q \equiv 3 \mod 4$	2	$\langle\phi, \gamma\rangle$
$4 \circ 2^{\cdot}\Omega^+_{10}(q).4$	$q \equiv 5 \mod 8$	4	$\langle\phi, \gamma\rangle$
$8 \circ 2^{\cdot}\Omega^+_{10}(q).2$	$q \equiv 9 \mod 16$	4	$\langle\delta^4, \phi, \gamma\rangle$
$16 \circ 2^{\cdot}\Omega^+_{10}(q)$	$q \equiv 1 \mod 16$	4	$\langle\delta^4, \phi, \gamma\rangle$

7.2 $\mathrm{SU}_{16}(q)$

Table 7.3: The maximal subgroups of $\mathrm{SU}_{16}(q)$ of geometric type
$d := |Z(\mathrm{SU}_{16}(q))| = (q+1, 16), |\delta| = d, |\phi| = 2e, \phi^e = \gamma, q = p^e$

Class	Group	Notes	Classes	Stabiliser
1	$E_q^{29} : \mathrm{SU}_{14}(q).(q^2-1)$		1	$\langle \delta, \phi \rangle$
1	$E_q^{52} : (\mathrm{SL}_2(q^2) \times \mathrm{SU}_{12}(q)).(q^2-1)$		1	$\langle \delta, \phi \rangle$
1	$E_q^{69} : (\mathrm{SL}_3(q^2) \times \mathrm{SU}_{10}(q)).(q^2-1)$		1	$\langle \delta, \phi \rangle$
1	$E_q^{80} : (\mathrm{SL}_4(q^2) \times \mathrm{SU}_8(q)).(q^2-1)$		1	$\langle \delta, \phi \rangle$
1	$E_q^{85} : (\mathrm{SL}_5(q^2) \times \mathrm{SU}_6(q)).(q^2-1)$		1	$\langle \delta, \phi \rangle$
1	$E_q^{84} : (\mathrm{SL}_6(q^2) \times \mathrm{SU}_4(q)).(q^2-1)$		1	$\langle \delta, \phi \rangle$
1	$E_q^{77} : (\mathrm{SL}_7(q^2) \times \mathrm{SU}_2(q)).(q^2-1)$		1	$\langle \delta, \phi \rangle$
1	$E_q^{64} : \mathrm{SL}_8(q^2).(q-1)$		1	$\langle \delta, \phi \rangle$
1	$\mathrm{SU}_{15}(q).(q+1)$		1	$\langle \delta, \phi \rangle$
1	$(\mathrm{SU}_2(q) \times \mathrm{SU}_{14}(q)).(q+1)$		1	$\langle \delta, \phi \rangle$
1	$(\mathrm{SU}_3(q) \times \mathrm{SU}_{13}(q)).(q+1)$		1	$\langle \delta, \phi \rangle$
1	$(\mathrm{SU}_4(q) \times \mathrm{SU}_{12}(q)).(q+1)$		1	$\langle \delta, \phi \rangle$
1	$(\mathrm{SU}_5(q) \times \mathrm{SU}_{11}(q)).(q+1)$		1	$\langle \delta, \phi \rangle$
1	$(\mathrm{SU}_6(q) \times \mathrm{SU}_{10}(q)).(q+1)$		1	$\langle \delta, \phi \rangle$
1	$(\mathrm{SU}_7(q) \times \mathrm{SU}_9(q)).(q+1)$		1	$\langle \delta, \phi \rangle$
2	$\mathrm{SU}_8(q)^2.(q+1).2$		1	$\langle \delta, \phi \rangle$
2	$\mathrm{SU}_4(q)^4.(q+1)^3.S_4$		1	$\langle \delta, \phi \rangle$
2	$\mathrm{SU}_2(q)^8.(q+1)^7.S_8$	$q \neq 2$	1	$\langle \delta, \phi \rangle$
2	$(q+1)^{15}.S_{16}$		1	$\langle \delta, \phi \rangle$
2	$\mathrm{SL}_8(q^2).(q-1).2$		1	$\langle \delta, \phi \rangle$
4	$(\mathrm{SU}_2(q) \circ \mathrm{SU}_8(q)).[(q+1,2)^2]$	$q \neq 2$	$(q+1,2)$	$\langle \delta^c, \phi \rangle$
5	$\mathrm{SU}_{16}(q_0).\left[\left(\frac{q+1}{q_0+1}, 16\right)\right]$	$q = q_0^r, r \geq 3$ prime	$\left(\frac{q+1}{q_0+1}, 16\right)$	$\langle \delta^c, \phi \rangle$
5	$\mathrm{SO}_{16}^{+}(q).[(q+1,16)]$	q odd	$\frac{(q+1,16)}{2}$	$\langle \delta^c, \phi \rangle$
5	$\mathrm{SO}_{16}^{-}(q).[(q+1,16)]$	q odd	$\frac{(q+1,16)}{2}$	$\langle \delta^c, \phi\delta^{\frac{p-1}{2}} \rangle$
5	$\mathrm{Sp}_{16}(q).[(q+1,8)]$		$(q+1,8)$	$\langle \delta^c, \phi \rangle$
6	$((q+1,16) \circ 2^{1+8}).\mathrm{Sp}_8(2)$	$q = p \equiv 3 \mod 4$	$(q+1,16)$	$\langle \delta^c, \phi \rangle$
7	$(q+1,4).\mathrm{U}_4(q)^2.\left[(q+1,4)^2\right].2$		$(q+1,4)$	$\langle \delta^c, \phi \rangle$

Table 7.4: The maximal subgroups of $\mathrm{SU}_{16}(q)$ in class $\mathcal{S}$

$d := |Z(\mathrm{SU}_{16}(q))| = (q+1,16), |\delta| = d, |\phi| = 2e, \phi^e = \gamma, q = p^e$

Group	Notes	Classes	Stabiliser
$d \circ 2^{\cdot}A_{11}$	$q = p \equiv 2,6,7,8,10 \mod 11$	d	$\langle\gamma\rangle$
$8 \circ 4^{\cdot}M_{22}$	$q = 7$	8	$\langle\gamma\rangle$
$d \times M_{12}$	$q = p \equiv 17,29,35,41,65 \mod 66$	d	$\langle\gamma\rangle$
$d \times M_{12}$	$q = p \equiv 7,13,19,43,61 \mod 66$	d	$\langle\gamma\delta\rangle$
$4^{\cdot}_2\mathrm{L}_3(4).2_2$	$q = p = 3$, N1	4	$\langle\gamma\rangle$
$d \circ \mathrm{SL}_2(31)$	$q = p$ non-square mod 31, $p \equiv \pm1 \mod 8$	d	$\langle\gamma\rangle$
$d \circ \mathrm{SL}_2(31)$	$q = p$ non-square mod 31, $p \equiv \pm3 \mod 8$	d	$\langle\gamma\delta\rangle$
$d \times \mathrm{L}_3(3)$	$q = p \equiv 17,23,77 \mod 78$	d	$\langle\gamma\rangle$
$d \times \mathrm{L}_3(3)$	$q = p \equiv 17,23,77 \mod 78$	d	$\langle\gamma\rangle$
$d \times \mathrm{L}_3(3)$	$q = p \equiv 25,43,49 \mod 78$	d	$\langle\gamma\delta\rangle$
$d \times \mathrm{L}_3(3)$	$q = p \equiv 25,43,49 \mod 78$	d	$\langle\gamma\delta\rangle$
$d \times \mathrm{L}_3(3)$	$q = p^2$, $p \equiv 2,5,6,7,8,11 \mod 13$	$2d$	$\langle\gamma\delta\rangle$
$\mathrm{SU}_4(q).(q+1,4)$	$q = 3^e$	d	$\langle\phi\rangle$
$\mathrm{L}_4(q^2).2$	q even	1	$\langle\phi\rangle$
$2 \times (\mathrm{L}_4(q^2).(4 \times 2))$	$q \equiv 1 \mod 4$	2	$\langle\phi\rangle$
$4 \circ 2^{\cdot}\mathrm{L}_4(q^2).(4 \times 2)$	$q \equiv 3 \mod 8$	4	$\langle\phi\rangle$
$8 \circ 2^{\cdot}\mathrm{L}_4(q^2).2^2$	$q \equiv 7 \mod 16$	4	$\langle\delta^4, \phi\rangle$
$16 \circ 2^{\cdot}\mathrm{L}_4(q^2).2$	$q \equiv 15 \mod 16$	4	$\langle\delta^4, \phi\rangle$
$\Omega^-_{10}(q)$	q even	1	$\langle\phi\rangle$
$2^{\cdot}\Omega^-_{10}(q).2$	$q \equiv 1 \mod 4$	2	$\langle\phi\rangle$
$4 \circ 2^{\cdot}\Omega^-_{10}(q).4$	$q \equiv 3 \mod 8$	4	$\langle\phi\rangle$
$8 \circ 2^{\cdot}\Omega^-_{10}(q).2$	$q \equiv 7 \mod 16$	4	$\langle\delta^4, \phi\rangle$
$16 \circ 2^{\cdot}\Omega^-_{10}(q)$	$q \equiv 15 \mod 16$	4	$\langle\delta^4, \phi\rangle$

N1 - Maximal under subgroups containing $\langle\gamma\rangle$.

7.3 $\mathrm{Sp}_{16}(q)$

Table 7.5: The maximal subgroups of $\mathrm{Sp}_{16}(q)$ of geometric type
$d := |Z(\mathrm{Sp}_{16}(q))| = (q-1,2), |\delta| = d, |\phi| = e, q = p^e$

Class	Group	Notes	Classes	Stabiliser
1	$E_q^{15} : ((q-1) \times \mathrm{Sp}_{14}(q))$		1	$\langle\delta,\phi\rangle$
1	$E_q^{27} : (\mathrm{GL}_2(q) \times \mathrm{Sp}_{12}(q))$		1	$\langle\delta,\phi\rangle$
1	$E_q^{36} : (\mathrm{GL}_3(q) \times \mathrm{Sp}_{10}(q))$		1	$\langle\delta,\phi\rangle$
1	$E_q^{42} : (\mathrm{GL}_4(q) \times \mathrm{Sp}_{8}(q))$		1	$\langle\delta,\phi\rangle$
1	$E_q^{45} : (\mathrm{GL}_5(q) \times \mathrm{Sp}_{6}(q))$		1	$\langle\delta,\phi\rangle$
1	$E_q^{45} : (\mathrm{GL}_6(q) \times \mathrm{Sp}_{4}(q))$		1	$\langle\delta,\phi\rangle$
1	$E_q^{42} : (\mathrm{GL}_7(q) \times \mathrm{Sp}_{2}(q))$		1	$\langle\delta,\phi\rangle$
1	$E_q^{36} : \mathrm{GL}_8(q)$		1	$\langle\delta,\phi\rangle$
1	$\mathrm{Sp}_2(q) \times \mathrm{Sp}_{14}(q)$		1	$\langle\delta,\phi\rangle$
1	$\mathrm{Sp}_4(q) \times \mathrm{Sp}_{12}(q)$		1	$\langle\delta,\phi\rangle$
1	$\mathrm{Sp}_6(q) \times \mathrm{Sp}_{10}(q)$		1	$\langle\delta,\phi\rangle$
2	$\mathrm{Sp}_8(q)^2 : 2$		1	$\langle\delta,\phi\rangle$
2	$\mathrm{Sp}_4(q)^4 : S_4$		1	$\langle\delta,\phi\rangle$
2	$\mathrm{Sp}_2(q)^8 : S_8$	$q \neq 2$	1	$\langle\delta,\phi\rangle$
2	$\mathrm{GL}_8(q).2$	$p \neq 2$	1	$\langle\delta,\phi\rangle$
3	$\mathrm{Sp}_8(q^2).2$		1	$\langle\delta,\phi\rangle$
3	$\mathrm{GU}_8(q).2$	$p \neq 2$	1	$\langle\delta,\phi\rangle$
4	$(\mathrm{Sp}_4(q) \circ \mathrm{GO}_4^+(q)).2$	$p \neq 2$	1	$\langle\delta,\phi\rangle$
4	$(\mathrm{Sp}_4(q) \circ \mathrm{GO}_4^-(q)).2$	$p \neq 2$	1	$\langle\delta,\phi\rangle$
4	$(\mathrm{Sp}_2(q) \circ \mathrm{GO}_8^+(q)).2$	$p \neq 2$	1	$\langle\delta,\phi\rangle$
4	$(\mathrm{Sp}_2(q) \circ \mathrm{GO}_8^-(q)).2$	$p \neq 2$	1	$\langle\delta,\phi\rangle$
5	$\mathrm{Sp}_{16}(q_0).2$	$q = q_0^2$, q odd	2	$\langle\phi\rangle$
5	$\mathrm{Sp}_{16}(q_0)$	$q = q_0^r$, r odd prime, q odd	1	$\langle\delta,\phi\rangle$
5	$\mathrm{Sp}_{16}(q_0)$	$q = q_0^r$, r prime, q even	1	$\langle\delta,\phi\rangle$
6	$2_-^{1+8}.\mathrm{SO}_8^-(2)$	$q = p \equiv \pm 1 \mod 8$	2	$\langle\phi\rangle$
6	$2_-^{1+8}.\Omega_8^-(2)$	$q = p \equiv \pm 3 \mod 8$	1	$\langle\delta,\phi\rangle$
8	$\mathrm{GO}_{16}^+(q)$	q even	1	$\langle\delta,\phi\rangle$
8	$\mathrm{GO}_{16}^-(q)$	q even	1	$\langle\delta,\phi\rangle$

Table 7.6: The maximal subgroups of $\mathrm{Sp}_{16}(q)$ in class $\mathcal{S}$

$d := |Z(\mathrm{Sp}_{16}(q))| = (q-1,2), |\delta| = d, |\phi| = e, q = p^e$

Group	Notes	Classes	Stabiliser
$A_{18}.2$	$q = p = 2$	1	1
$2^{\cdot}A_8$	$q = p = 7$	1	$\langle\delta\rangle$
$2^{\cdot}\mathrm{L}_2(17).2$	$q = p \equiv \pm1 \mod 12$	2	1
$2^{\cdot}\mathrm{L}_2(17)$	$q = p \equiv \pm5 \mod 12$, $p \neq 17$	1	$\langle\delta\rangle$
$2^{\cdot}\mathrm{L}_2(17).2$	$q = p \equiv \pm1 \mod 36$	2	1
$2^{\cdot}\mathrm{L}_2(17).2$	$q = p \equiv \pm1 \mod 36$	2	1
$2^{\cdot}\mathrm{L}_2(17).2$	$q = p \equiv \pm1 \mod 36$	2	1
$2^{\cdot}\mathrm{L}_2(17).2$	$q = p^3$, $p \equiv \pm11, \pm13 \mod 36$	6	1
$2^{\cdot}\mathrm{L}_2(17)$	$q = p \equiv \pm17 \mod 36$, $p \neq 17$	1	$\langle\delta\rangle$
$2^{\cdot}\mathrm{L}_2(17)$	$q = p \equiv \pm17 \mod 36$, $p \neq 17$	1	$\langle\delta\rangle$
$2^{\cdot}\mathrm{L}_2(17)$	$q = p \equiv \pm17 \mod 36$, $p \neq 17$	1	$\langle\delta\rangle$
$2^{\cdot}\mathrm{L}_2(17)$	$q = p^3$, $p \equiv \pm5, \pm7 \mod 36$	3	$\langle\delta\rangle$
$\mathrm{SL}_2(q)$	$p \geq 17$	1	$\langle\delta, \phi\rangle$
$\mathrm{Sp}_4(q)$	$p \neq 2, 5$	1	$\langle\delta, \phi\rangle$

7.4 $\Omega^+_{16}(q)$

Table 7.7: The maximal subgroups of $\Omega^+_{16}(q)$ of geometric type
$d := |Z(\Omega^+_{16}(q))| = (q-1,2), |\delta| = d, |\delta'| = d, |\gamma| = 2, |\phi| = e, q = p^e$

Class	Group	Notes	Classes	Stabiliser
1	$E_q^{14}.(\frac{q-1}{(q-1,2)} \times \Omega^+_{14}(q)).d$		1	$\langle\delta,\gamma,\delta',\phi\rangle$
1	$E_q^{25}.(\frac{1}{(q-1,2)}\mathrm{GL}_2(q) \times \Omega^+_{12}(q)).d$		1	$\langle\delta,\gamma,\delta',\phi\rangle$
1	$E_q^{33}.(\frac{1}{(q-1,2)}\mathrm{GL}_3(q) \times \Omega^+_{10}(q)).d$		1	$\langle\delta,\gamma,\delta',\phi\rangle$
1	$E_q^{38}.(\frac{1}{(q-1,2)}\mathrm{GL}_4(q) \times \Omega^+_{8}(q)).d$		1	$\langle\delta,\gamma,\delta',\phi\rangle$
1	$E_q^{40}.(\frac{1}{(q-1,2)}\mathrm{GL}_5(q) \times \Omega^+_{6}(q)).d$		1	$\langle\delta,\gamma,\delta',\phi\rangle$
1	$E_q^{39}.(\frac{1}{(q-1,2)}\mathrm{GL}_6(q) \times \Omega^+_{4}(q)).d$		1	$\langle\delta,\gamma,\delta',\phi\rangle$
1	$E_q^{35}.(\frac{1}{(q-1,2)}\mathrm{GL}_7(q) \times \Omega^+_{2}(q)).d$	N1	1	$\langle\delta,\gamma,\delta',\phi\rangle$
1	$E_q^{28}.\frac{1}{(q-1,2)}\mathrm{GL}_8(q)$		2	$\langle\delta,\delta',\phi\rangle$
1	$\Omega^\circ_{15}(q).2$	q odd	2	$\langle\gamma,\delta',\phi\rangle$
1	$(\Omega^+_2(q) \times \Omega^+_{14}(q)).2^d$	$q \neq 2$, N2 if $q = 3$	1	$\langle\delta,\gamma,\delta',\phi\rangle$
1	$(\Omega^-_2(q) \times \Omega^-_{14}(q)).2^d$		1	$\langle\delta,\gamma,\delta',\phi\rangle$
1	$(\Omega^\circ_3(q) \times \Omega^\circ_{13}(q)).2^2$	q odd	2	$\langle\gamma,\delta',\phi\rangle$
1	$(\Omega^+_4(q) \times \Omega^+_{12}(q)).2^d$		1	$\langle\delta,\gamma,\delta',\phi\rangle$
1	$(\Omega^-_4(q) \times \Omega^-_{12}(q)).2^d$		1	$\langle\delta,\gamma,\delta',\phi\rangle$
1	$(\Omega^\circ_5(q) \times \Omega^\circ_{11}(q)).2^2$	q odd	2	$\langle\gamma,\delta',\phi\rangle$
1	$(\Omega^+_6(q) \times \Omega^+_{10}(q)).2^d$		1	$\langle\delta,\gamma,\delta',\phi\rangle$
1	$(\Omega^-_6(q) \times \Omega^-_{10}(q)).2^d$		1	$\langle\delta,\gamma,\delta',\phi\rangle$
1	$(\Omega^\circ_7(q) \times \Omega^\circ_9(q)).2^2$	q odd	2	$\langle\gamma,\delta',\phi\rangle$
1	$\mathrm{Sp}_{14}(q)$	q even	1	$\langle\gamma,\phi\rangle$
2	$\Omega^+_8(q)^2.2^d.2$		1	$\langle\delta,\gamma,\delta',\phi\rangle$
2	$\Omega^-_8(q)^2.2^d.2$		1	$\langle\delta,\gamma,\delta',\phi\rangle$
2	$\Omega^+_4(q)^4.2^{3d}.S_4$	$q \neq 2$	1	$\langle\delta,\gamma,\delta',\phi\rangle$
2	$\Omega^-_4(q)^4.2^{3d}.S_4$		1	$\langle\delta,\gamma,\delta',\phi\rangle$
2	$\Omega^+_2(q)^8.2^{7d}.S_8$	$q \neq 2,3,4$, N2 if $q = 5$	1	$\langle\delta,\gamma,\delta',\phi\rangle$
2	$\Omega^-_2(q)^8.2^{7d}.S_8$	N2 if $q = 3$	1	$\langle\delta,\gamma,\delta',\phi\rangle$
2	$2^{15}.S_{16}$	$q = p \equiv \pm 1 \mod 8$	4	$\langle\gamma\rangle$
2	$2^{15}.A_{16}$	$q = p \equiv \pm 3 \mod 8$	2	$\langle\gamma,\delta'\rangle$
2	$\mathrm{SL}_8(q).\frac{q-1}{d}.2$		2	$\langle\delta,\delta',\phi\rangle$
3	$((q+1) \circ \mathrm{SU}_8(q)).[(2,q)(8,q+1)]$		2	$\langle\delta,\delta',\phi\rangle$
3	$\Omega^+_8(q^2).2^2$		2	$\langle\delta,\delta',\phi\rangle$
4	$(\mathrm{Sp}_2(q) \circ \mathrm{Sp}_8(q)).d$	$q \neq 2$	$2d$	$\langle\delta,\phi\rangle$
4	$(\mathrm{SO}^+_4(q) \circ \mathrm{SO}^-_4(q)).2^2$	$p \neq 2$, N3	2	$\langle\delta,\delta',\phi\rangle$
5	$\Omega^+_{16}(q_0)$	$q = q_0^r$, r prime, r odd or q even	1	$\langle\delta,\gamma,\delta',\phi\rangle$
5	$\mathrm{SO}^+_{16}(q_0).2$	$q = q_0^2$, q odd	4	$\langle\gamma,\phi\rangle$
5	$\Omega^-_{16}(q_0)$	$q = q_0^2$, q even	1	$\langle\gamma,\phi\rangle$
5	$\mathrm{SO}^-_{16}(q_0)$	$q = q_0^2$, q odd	2	$\langle\gamma,\delta',\phi\rangle$
6	$2^{1+8}_+.\Omega^+_8(2)$	$q = p \equiv 3 \mod 8$	4	$\langle\delta\rangle$
6	$2^{1+8}_+.\mathrm{SO}^+_8(2)$	$q = p \equiv 1 \mod 8$	8	1
7	$d.\mathrm{S}_2(q)^4.d^3.\mathrm{S}_4$	$p \neq 2, q \neq 3$	4	$\langle\delta,\phi\rangle$
7	$d.\mathrm{S}_4(q)^2.d.2$	$p \neq 2$	4	$\langle\delta,\phi\rangle$
7	$2.\mathrm{PSO}^-_4(q)^2.2^3$	$p \neq 2$	2	$\langle\delta,\delta',\phi\rangle$

N1 - Maximal under subgroups not contained in $\langle \delta, \delta', \phi \rangle$.
N2 - Maximal under subgroups not contained in $\langle \delta', \gamma, \phi \rangle$.
N3 - Extend by $B \times \langle \phi^i \rangle$ for some integer i and $B < \langle \delta, \delta', \gamma \rangle$. Then:
• there is one class of novelty maximal subgroups if $B = \langle \delta \rangle$;
• there are two classes of novelty maximal subgroups if $B = \langle \delta' \rangle$;
• there are no classes of maximal subgroups otherwise.

Table 7.8: The maximal subgroups of $\Omega^+_{16}(q)$ in class $\mathcal{S}$

$d := |Z(\Omega^+_{16}(q))| = (q-1,2), |\delta| = d, |\delta'| = d, |\gamma| = 2, |\phi| = e, q = p^e$

Group	Notes	Classes	Stabiliser
$d \times A_{17}$	$q = p \equiv \pm1, \pm2, \pm4, \pm8 \mod 17$	d^2	$\langle \gamma \rangle$
$2^{\cdot}A_{11}.2$	$q = 11$	8	1
$2^{\cdot}A_{10}.2$	$q = p \equiv \pm1 \mod 10$, $p \neq 11$	8	1
$2^{\cdot}A_{10}$	$q = p \equiv \pm3 \mod 10$	4	$\langle \delta \rangle$
$2 \times M_{12}$	$q = 11$	4	$\langle \gamma \rangle$
$2^{\cdot}\mathrm{Sz}(8)$	$q = 13$	8	1
$2 \times \mathrm{L}_3(3)$	$q = 13$	4	$\langle \gamma \rangle$
$d \times \mathrm{L}_2(17)$	$q = p \equiv \pm1, \pm2, \pm4, \pm8 \mod 17$, N1	d^2	$\langle \gamma \rangle$
$2 \times \mathrm{L}_2(17)$	$q = p \equiv \pm1, \pm2, \pm4, \pm8 \mod 17$, $p \equiv \pm1 \mod 9$	4	$\langle \gamma \rangle$
$2 \times \mathrm{L}_2(17)$	$q = p \equiv \pm1, \pm2, \pm4, \pm8 \mod 17$, $p \equiv \pm1 \mod 9$	4	$\langle \gamma \rangle$
$2 \times \mathrm{L}_2(17)$	$q = p \equiv \pm1, \pm2, \pm4, \pm8 \mod 17$, $p \equiv \pm1 \mod 9$	4	$\langle \gamma \rangle$
$d \times \mathrm{L}_2(17)$	$q = p^3, p \equiv \pm1, \pm2, \pm4, \pm8 \mod 17$, $p \equiv \pm2, \pm4 \mod 9$	$3d^2$	$\langle \gamma \rangle$
$2 \times \mathrm{L}_2(q^4).4^*$	$q \neq 2$, C1, C2	$2d$	$\langle \delta, \phi \rangle^*$
$2 \times \mathrm{S}_4(q^2).2^*$	$p \neq 2$, C1, C2	4	$\langle \delta, \phi \rangle^*$
$\mathrm{Sp}_8(q)$	$p = 2$	1	$\langle \phi \rangle$
$2^{\cdot}\Omega^\circ_9(q)$	$p \neq 2$	4	$\langle \delta, \phi \rangle$

* denotes a result which is currently conjectured
N1 - Maximal under subgroups containing $\langle \gamma \rangle$
C1 - Structure of the group depends on Conjecture 2.3.3
C2 - Action of the field automorphism currently conjectured - see Remark 4.3.12

7.5 $\Omega^-_{16}(q)$

Table 7.9: The maximal subgroups of $\Omega^-_{16}(q)$ of geometric type
$|Z(\Omega^-_{16}(q))| = 1, |\delta| = (q-1,2), |\gamma| = 2, |\varphi| = 2e, \varphi^e = \gamma, q = p^e$

Class	Group	Notes	Classes	Stabiliser
1	$E_q^{14} : (\frac{q-1}{(q-1,2)} \times \Omega^-_{14}(q)).(q-1,2)$		1	$\langle\gamma,\delta,\varphi\rangle$
1	$E_q^{25} : (\frac{1}{(q-1,2)}\mathrm{GL}_2(q) \times \Omega^-_{12}(q)).(q-1,2)$		1	$\langle\gamma,\delta,\varphi\rangle$
1	$E_q^{33} : (\frac{1}{(q-1,2)}\mathrm{GL}_3(q) \times \Omega^-_{10}(q)).(q-1,2)$		1	$\langle\gamma,\delta,\varphi\rangle$
1	$E_q^{38} : (\frac{1}{(q-1,2)}\mathrm{GL}_4(q) \times \Omega^-_{8}(q)).(q-1,2)$		1	$\langle\gamma,\delta,\varphi\rangle$
1	$E_q^{40} : (\frac{1}{(q-1,2)}\mathrm{GL}_5(q) \times \Omega^-_{6}(q)).(q-1,2)$		1	$\langle\gamma,\delta,\varphi\rangle$
1	$E_q^{39} : (\frac{1}{(q-1,2)}\mathrm{GL}_6(q) \times \Omega^-_{4}(q)).(q-1,2)$		1	$\langle\gamma,\delta,\varphi\rangle$
1	$E_q^{35} : (\frac{1}{(q-1,2)}\mathrm{GL}_7(q) \times \Omega^-_{2}(q)).(q-1,2)$		1	$\langle\gamma,\delta,\varphi\rangle$
1	$\Omega^\circ_{15}(q).2$	q odd	2	$\langle\gamma,\varphi\rangle$
1	$(\Omega^+_2(q) \times \Omega^-_{14}(q)).2^{(q-1,2)}$	$q \neq 2$, N1 if $q = 3$	1	$\langle\gamma,\delta,\varphi\rangle$
1	$(\Omega^-_2(q) \times \Omega^+_{14}(q)).2^{(q-1,2)}$		1	$\langle\gamma,\delta,\varphi\rangle$
1	$(\Omega^\circ_3(q) \times \Omega^\circ_{13}(q)).2^2$	q odd	2	$\langle\gamma,\varphi\rangle$
1	$(\Omega^+_4(q) \times \Omega^-_{12}(q)).2^{(q-1,2)}$		1	$\langle\gamma,\delta,\varphi\rangle$
1	$(\Omega^-_4(q) \times \Omega^+_{12}(q)).2^{(q-1,2)}$		1	$\langle\gamma,\delta,\varphi\rangle$
1	$(\Omega^\circ_5(q) \times \Omega^\circ_{11}(q)).2^2$	q odd	2	$\langle\gamma,\varphi\rangle$
1	$(\Omega^+_6(q) \times \Omega^-_{10}(q)).2^{(q-1,2)}$		1	$\langle\gamma,\delta,\varphi\rangle$
1	$(\Omega^-_6(q) \times \Omega^+_{10}(q)).2^{(q-1,2)}$		1	$\langle\gamma,\delta,\varphi\rangle$
1	$(\Omega^\circ_7(q) \times \Omega^\circ_9(q)).2^2$	q odd	2	$\langle\gamma,\varphi\rangle$
1	$(\Omega^+_8(q) \times \Omega^-_8(q)).2^{(q-1,2)}$		1	$\langle\gamma,\delta,\varphi\rangle$
1	$\mathrm{Sp}_{14}(q)$	q even	1	$\langle\gamma,\varphi\rangle$
3	$\Omega^-_8(q^2).2$		1	$\langle\gamma,\delta,\varphi\rangle$
5	$\Omega^-_{16}(q_0)$	$q = q_0^r$, r odd prime	1	$\langle\gamma,\delta,\varphi\rangle$

N1 - Maximal under extensions not containing $\langle\gamma,\varphi\rangle$.

Table 7.10: The maximal subgroups of $\Omega_{16}^{-}(q)$ in class $\mathcal{S}$

$|Z(\Omega_{16}^{-}(q))| = 1, |\delta| = (q-1,2), |\gamma| = 2, |\varphi| = 2e, \varphi^e = \gamma, q = p^e$

Group	Notes	Classes	Stabiliser
A_{18}	$q = p = 3$	2	$\langle\gamma\rangle$
A_{17}	$q = p \equiv \pm 3, \pm 5, \pm 6, \pm 7 \mod 17$, $p \neq 3$	2	$\langle\gamma\rangle$
$\mathrm{L}_2(17)$	$q = p \equiv \pm 3, \pm 5, \pm 6, \pm 7 \mod 17$, $p \neq 3$	2	$\langle\gamma\rangle$
$\mathrm{L}_2(17)$	$q = p \equiv \pm 3, \pm 5, \pm 6, \pm 7 \mod 17$, $p \equiv \pm 1 \mod 9$	2	$\langle\gamma\rangle$
$\mathrm{L}_2(17)$	$q = p \equiv \pm 3, \pm 5, \pm 6, \pm 7 \mod 17$, $p \equiv \pm 1 \mod 9$	2	$\langle\gamma\rangle$
$\mathrm{L}_2(17)$	$q = p \equiv \pm 3, \pm 5, \pm 6, \pm 7 \mod 17$, $p \equiv \pm 1 \mod 9$	2	$\langle\gamma\rangle$
$\mathrm{L}_2(17)$	$q = p^3$, $p \equiv \pm 3, \pm 5, \pm 6, \pm 7 \mod 17$, $p \equiv \pm 2, \pm 4 \mod 9$	6	$\langle\gamma\rangle$

7.6 $\mathrm{SL}_{17}(q)$

Table 7.11: The maximal subgroups of $\mathrm{SL}_{17}(q)$ of geometric type
$d := |Z(\mathrm{SL}_{17}(q))| = (q-1,17), |\delta| = d, |\phi| = e, |\gamma| = 2, q = p^e$

Class	Group	Notes	Classes	Stabiliser
1	$E_q^{16} : \mathrm{GL}_{16}(q)$		2	$\langle\delta,\phi\rangle$
1	$E_q^{30} : (\mathrm{SL}_2(q)\times\mathrm{SL}_{15}(q)) : (q-1)$		2	$\langle\delta,\phi\rangle$
1	$E_q^{42} : (\mathrm{SL}_3(q)\times\mathrm{SL}_{14}(q)) : (q-1)$		2	$\langle\delta,\phi\rangle$
1	$E_q^{52} : (\mathrm{SL}_4(q)\times\mathrm{SL}_{13}(q)) : (q-1)$		2	$\langle\delta,\phi\rangle$
1	$E_q^{60} : (\mathrm{SL}_5(q)\times\mathrm{SL}_{12}(q)) : (q-1)$		2	$\langle\delta,\phi\rangle$
1	$E_q^{66} : (\mathrm{SL}_6(q)\times\mathrm{SL}_{11}(q)) : (q-1)$		2	$\langle\delta,\phi\rangle$
1	$E_q^{70} : (\mathrm{SL}_7(q)\times\mathrm{SL}_{10}(q)) : (q-1)$		2	$\langle\delta,\phi\rangle$
1	$E_q^{72} : (\mathrm{SL}_8(q)\times\mathrm{SL}_9(q)) : (q-1)$		2	$\langle\delta,\phi\rangle$
1	$\mathrm{GL}_{16}(q)$	N1	1	$\langle\delta,\phi,\gamma\rangle$
1	$(\mathrm{SL}_2(q)\times\mathrm{SL}_{15}(q)) : (q-1)$	N1	1	$\langle\delta,\phi,\gamma\rangle$
1	$(\mathrm{SL}_3(q)\times\mathrm{SL}_{14}(q)) : (q-1)$	N1	1	$\langle\delta,\phi,\gamma\rangle$
1	$(\mathrm{SL}_4(q)\times\mathrm{SL}_{13}(q)) : (q-1)$	N1	1	$\langle\delta,\phi,\gamma\rangle$
1	$(\mathrm{SL}_5(q)\times\mathrm{SL}_{12}(q)) : (q-1)$	N1	1	$\langle\delta,\phi,\gamma\rangle$
1	$(\mathrm{SL}_6(q)\times\mathrm{SL}_{11}(q)) : (q-1)$	N1	1	$\langle\delta,\phi,\gamma\rangle$
1	$(\mathrm{SL}_7(q)\times\mathrm{SL}_{10}(q)) : (q-1)$	N1	1	$\langle\delta,\phi,\gamma\rangle$
1	$(\mathrm{SL}_8(q)\times\mathrm{SL}_9(q)) : (q-1)$	N1	1	$\langle\delta,\phi,\gamma\rangle$
1	$E_q^{31} : \mathrm{SL}_{15}(q) : (q-1)^2$	N1	1	$\langle\delta,\phi,\gamma\rangle$
1	$E_q^{56} : (\mathrm{SL}_2(q)^2\times\mathrm{SL}_{13}(q)) : (q-1)^2$	N1	1	$\langle\delta,\phi,\gamma\rangle$
1	$E_q^{75} : (\mathrm{SL}_3(q)^2\times\mathrm{SL}_{11}(q)) : (q-1)^2$	N1	1	$\langle\delta,\phi,\gamma\rangle$
1	$E_q^{88} : (\mathrm{SL}_4(q)^2\times\mathrm{SL}_9(q)) : (q-1)^2$	N1	1	$\langle\delta,\phi,\gamma\rangle$
1	$E_q^{95} : (\mathrm{SL}_5(q)^2\times\mathrm{SL}_7(q)) : (q-1)^2$	N1	1	$\langle\delta,\phi,\gamma\rangle$
1	$E_q^{96} : (\mathrm{SL}_6(q)^2\times\mathrm{SL}_5(q)) : (q-1)^2$	N1	1	$\langle\delta,\phi,\gamma\rangle$
1	$E_q^{91} : (\mathrm{SL}_7(q)^2\times\mathrm{SL}_3(q)) : (q-1)^2$	N1	1	$\langle\delta,\phi,\gamma\rangle$
1	$E_q^{80} : \mathrm{SL}_8(q)^2 : (q-1)^2$	N1	1	$\langle\delta,\phi,\gamma\rangle$
2	$(q-1)^{16}.S_{17}$	$q \geq 5$	1	$\langle\delta,\phi,\gamma\rangle$
3	$\frac{q^{17}-1}{q-1}.17$		1	$\langle\delta,\phi,\gamma\rangle$
5	$\mathrm{SL}_{17}(q_0).c$	$q = q_0^r$, r prime	$\left(\frac{q-1}{q_0-1}, 17\right)$	$\langle\delta^c,\phi,\gamma\rangle$
6	$(d \circ 17^{1+2}).\mathrm{Sp}_2(17)$	$q = p \equiv 1 \mod 17$	d	$\langle\delta^c,\phi,\gamma\rangle$
8	$\mathrm{SO}^\circ_{17}(q).(q-1,17)$	q odd	d	$\langle\delta^c,\phi,\gamma\rangle$
8	$\mathrm{SU}_{17}(q_0).(q_0-1,17)$	$q = q_0^2$	$(q_0-1,17)$	$\langle\delta^c,\phi,\gamma\rangle$

N1 - Maximal under subgroups not contained in $\langle\delta,\phi\rangle$

There are no maximal subgroups in class $\mathcal{S}$ for $\Omega = \mathrm{SL}_{17}(q)$.

7.7 $\mathrm{SU}_{17}(q)$

Table 7.12: The maximal subgroups of $\mathrm{SU}_{17}(q)$ of geometric type
$d := |Z(\mathrm{SU}_{17}(q))| = (q+1, 17), |\delta| = d, |\phi| = 2e, \phi^e = \gamma, q = p^e$

Class	Group	Notes	Classes	Stabiliser
1	$E_q^{31} : \mathrm{SU}_{15}(q).(q^2-1)$		1	$\langle \delta, \phi \rangle$
1	$E_q^{56} : (\mathrm{SL}_2(q^2) \times \mathrm{SU}_{13}(q)).(q^2-1)$		1	$\langle \delta, \phi \rangle$
1	$E_q^{75} : (\mathrm{SL}_3(q^2) \times \mathrm{SU}_{11}(q)).(q^2-1)$		1	$\langle \delta, \phi \rangle$
1	$E_q^{88} : (\mathrm{SL}_4(q^2) \times \mathrm{SU}_9(q)).(q^2-1)$		1	$\langle \delta, \phi \rangle$
1	$E_q^{95} : (\mathrm{SL}_5(q^2) \times \mathrm{SU}_7(q)).(q^2-1)$		1	$\langle \delta, \phi \rangle$
1	$E_q^{96} : (\mathrm{SL}_6(q^2) \times \mathrm{SU}_5(q)).(q^2-1)$		1	$\langle \delta, \phi \rangle$
1	$E_q^{91} : (\mathrm{SL}_7(q^2) \times \mathrm{SU}_3(q)).(q^2-1)$		1	$\langle \delta, \phi \rangle$
1	$E_q^{80} : \mathrm{SL}_8(q^2).(q^2-1)$		1	$\langle \delta, \phi \rangle$
1	$\mathrm{SU}_{16}(q).(q+1)$		1	$\langle \delta, \phi \rangle$
1	$(\mathrm{SU}_2(q) \times \mathrm{SU}_{15}(q)).(q+1)$		1	$\langle \delta, \phi \rangle$
1	$(\mathrm{SU}_3(q) \times \mathrm{SU}_{14}(q)).(q+1)$		1	$\langle \delta, \phi \rangle$
1	$(\mathrm{SU}_4(q) \times \mathrm{SU}_{13}(q)).(q+1)$		1	$\langle \delta, \phi \rangle$
1	$(\mathrm{SU}_5(q) \times \mathrm{SU}_{12}(q)).(q+1)$		1	$\langle \delta, \phi \rangle$
1	$(\mathrm{SU}_6(q) \times \mathrm{SU}_{11}(q)).(q+1)$		1	$\langle \delta, \phi \rangle$
1	$(\mathrm{SU}_7(q) \times \mathrm{SU}_{10}(q)).(q+1)$		1	$\langle \delta, \phi \rangle$
1	$(\mathrm{SU}_8(q) \times \mathrm{SU}_9(q)).(q+1)$		1	$\langle \delta, \phi \rangle$
2	$(q+1)^{16}.S_{17}$		1	$\langle \delta, \phi \rangle$
3	$\frac{q^{17}+1}{q+1}.17$		1	$\langle \delta, \phi \rangle$
5	$\mathrm{SU}_{17}(q_0).\left(\frac{q+1}{q_0+1}, 17\right)$	$q = q_0^r, r \geq 3$ prime	$\left(\frac{q+1}{q_0+1}, 17\right)$	$\langle \delta^c, \phi \rangle$
5	$\mathrm{SO}_{17}^{\circ}(q).(q+1, 17)$	q odd	d	$\langle \delta^c, \phi \rangle$
6	$(d \circ 17^{1+2}).\mathrm{Sp}_2(17)$	$q = p^f$	d	$\langle \delta^c, \phi \rangle$

$$f = \begin{cases} 1 \text{ if } p \equiv 16 \mod 17, \\ 2 \text{ if } p \equiv 4, 13 \mod 17, \\ 4 \text{ if } p \equiv 2, 8, 9, 15 \mod 17, \\ 8 \text{ if } p \equiv 3, 5, 6, 7, 10, 11, 12, 14 \mod 17. \end{cases}$$

There are no maximal subgroups in class $\mathcal{S}$ for $\Omega = \mathrm{SU}_{17}(q)$.

7.8 $\Omega_{17}^{\circ}(q)$

Table 7.13: The maximal subgroups of $\Omega_{17}^{\circ}(q)$ of geometric type
$|Z(\Omega_{17}^{\circ}(q))| = 1, |\delta| = 2, |\phi| = e, q = p^e, p \neq 2$

Class	Group	Notes	Classes	Stabiliser
1	$E_q^{15} : (\frac{q-1}{2} \times \Omega_{15}^{\circ}(q)).2$		1	$\langle \delta, \phi \rangle$
1	$E_q^{27} : (\frac{1}{2}\mathrm{GL}_2(q) \times \Omega_{13}^{\circ}(q)).2$		1	$\langle \delta, \phi \rangle$
1	$E_q^{36} : (\frac{1}{2}\mathrm{GL}_3(q) \times \Omega_{11}^{\circ}(q)).2$		1	$\langle \delta, \phi \rangle$
1	$E_q^{42} : (\frac{1}{2}\mathrm{GL}_4(q) \times \Omega_{9}^{\circ}(q)).2$		1	$\langle \delta, \phi \rangle$
1	$E_q^{45} : (\frac{1}{2}\mathrm{GL}_5(q) \times \Omega_{7}^{\circ}(q)).2$		1	$\langle \delta, \phi \rangle$
1	$E_q^{45} : (\frac{1}{2}\mathrm{GL}_6(q) \times \Omega_{5}^{\circ}(q)).2$		1	$\langle \delta, \phi \rangle$
1	$E_q^{42} : (\frac{1}{2}\mathrm{GL}_7(q) \times \Omega_{3}^{\circ}(q)).2$		1	$\langle \delta, \phi \rangle$
1	$E_q^{36} : \frac{1}{2}\mathrm{GL}_8(q)$		1	$\langle \delta, \phi \rangle$
1	$\Omega_{16}^{+}(q).2$		1	$\langle \delta, \phi \rangle$
1	$\Omega_{16}^{-}(q).2$		1	$\langle \delta, \phi \rangle$
1	$(\Omega_3^{\circ}(q) \times \Omega_{14}^{+}(q)).2^2$		1	$\langle \delta, \phi \rangle$
1	$(\Omega_3^{\circ}(q) \times \Omega_{14}^{-}(q)).2^2$		1	$\langle \delta, \phi \rangle$
1	$(\Omega_5^{\circ}(q) \times \Omega_{12}^{+}(q)).2^2$		1	$\langle \delta, \phi \rangle$
1	$(\Omega_5^{\circ}(q) \times \Omega_{12}^{-}(q)).2^2$		1	$\langle \delta, \phi \rangle$
1	$(\Omega_7^{\circ}(q) \times \Omega_{10}^{+}(q)).2^2$		1	$\langle \delta, \phi \rangle$
1	$(\Omega_7^{\circ}(q) \times \Omega_{10}^{-}(q)).2^2$		1	$\langle \delta, \phi \rangle$
1	$(\Omega_9^{\circ}(q) \times \Omega_{8}^{+}(q)).2^2$		1	$\langle \delta, \phi \rangle$
1	$(\Omega_9^{\circ}(q) \times \Omega_{8}^{-}(q)).2^2$		1	$\langle \delta, \phi \rangle$
1	$(\Omega_{11}^{\circ}(q) \times \Omega_{6}^{+}(q)).2^2$		1	$\langle \delta, \phi \rangle$
1	$(\Omega_{11}^{\circ}(q) \times \Omega_{6}^{-}(q)).2^2$		1	$\langle \delta, \phi \rangle$
1	$(\Omega_{13}^{\circ}(q) \times \Omega_{4}^{+}(q)).2^2$		1	$\langle \delta, \phi \rangle$
1	$(\Omega_{13}^{\circ}(q) \times \Omega_{4}^{-}(q)).2^2$		1	$\langle \delta, \phi \rangle$
1	$(\Omega_{15}^{\circ}(q) \times \Omega_{2}^{+}(q)).2^2$	$q \neq 3$	1	$\langle \delta, \phi \rangle$
1	$(\Omega_{15}^{\circ}(q) \times \Omega_{2}^{-}(q)).2^2$		1	$\langle \delta, \phi \rangle$
2	$2^{16}.A_{17}$	$p = q \equiv \pm 3 \mod 8$	1	$\langle \delta, \phi \rangle$
2	$2^{16}.S_{17}$	$p = q \equiv \pm 1 \mod 8$	2	$\langle \phi \rangle$
5	$\Omega_{17}^{\circ}(q_0)$	$q = q_0^r, r$ odd prime	1	$\langle \delta, \phi \rangle$
5	$\mathrm{SO}_{17}^{\circ}(q_0)$	$q = q_0^2$	2	$\langle \phi \rangle$

Table 7.14: The maximal subgroups of $\Omega^\circ_{17}(q)$ in class $\mathcal{S}$

$|Z(\Omega^\circ_{17}(q))| = 1, |\delta| = 2, |\phi| = e, q = p^e, p \neq 2$

Group	Notes	Classes	Stabiliser
$A_{19}.2$	$q = 19$	2	1
$A_{18}.2$	$q = p \neq 2, 3, 19$	2	1
$\mathrm{L}_2(16).4$	$q = p \equiv \pm 1 \mod 8$	2	1
$\mathrm{L}_2(16).2$	$q = p \equiv \pm 3 \mod 8$, $p \neq 3$	1	$\langle \delta \rangle$
$\mathrm{L}_2(16).2$	$q = p$, $p \equiv \pm 1 \mod 5$, $p \equiv \pm 1 \mod 8$	2	1
$\mathrm{L}_2(16).2$	$q = p$, $p \equiv \pm 1 \mod 5$, $p \equiv \pm 1 \mod 8$	2	1
$\mathrm{L}_2(16)$	$q = p$, $p \equiv \pm 1 \mod 5$, $p \equiv \pm 3 \mod 8$	1	$\langle \delta \rangle$
$\mathrm{L}_2(16)$	$q = p$, $p \equiv \pm 1 \mod 5$, $p \equiv \pm 3 \mod 8$	1	$\langle \delta \rangle$
$\mathrm{L}_2(16).2$	$q = p^2$, $p \equiv \pm 2 \mod 5$, $p \equiv \pm 1 \mod 8$, C1	2	$\langle \phi \rangle^*$
$\mathrm{L}_2(16).2$	$q = p^2$, $p \equiv \pm 2 \mod 5$, $p \equiv \pm 3 \mod 8$, C1	2	$\langle \phi\delta \rangle^*$
$\mathrm{L}_2(16)$	$q = p \equiv \pm 1 \mod 15$	2	1
$\mathrm{L}_2(16)$	$q = p^2$, $p \equiv \pm 4 \mod 15$	2	$\langle \phi \rangle$
$\mathrm{L}_2(16)$	$q = p^4$, $p \equiv \pm 2, \pm 7 \mod 15$	2	$\langle \phi \rangle$
$\mathrm{L}_2(q).2$	$p \geq 17$	2	$\langle \phi \rangle$

* denotes a result which is currently conjectured or incomplete

C1 - Class stabiliser currently conjectured - see Remark 3.3.10

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